The geometric genus of hypersurface singularities

Baldur Sigurðsson

June 24, 2014

Abstract

Using the path lattice cohomology we provide a conceptual topological characterization of the geometric genus for certain complex normal surface singularities with rational homology sphere links, which is uniformly valid for all superisolated and Newton nondegenerate hypersurface singularities. In this talk we will focus on the Newton nondegenerate case.

The content is to be published in JEMS in a joint article with Némethi András.

Definitions

- Let $(X,0)$ be a surface singularity, i.e. the germ of a two dimensional analytic space. We always assume that 0 is an *isolated* singularity of X.
- The *geometric genus* is the rank of the first cohomology of any resolution of X. That is, let $\tilde{X} \to X$ be a resolution; then $p_g = h^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$.
- Let M be the *link* of X. This means that given an embedding $(X,0) \hookrightarrow (\mathbb{C}^N, 0)$, we have $M = X \cap S_\epsilon^{2N-1}$ for $0 < \epsilon \ll 1$. We will always assume that $H_1(M, \mathbb{Q}) = 0$.
- We want to recover p_g from M.

Notation—the resolution graph

- Let $(\tilde{X}, E) \to (X, 0)$ be a good resolution. In particular, $\tilde{X}$ is a manifold and E is a normal crossing divisor. Denote by G the corresponding graph and $\mathcal{V}$ its vertex set. Then $E = \cup_{v \in \mathcal{V}} E_v$.
- By a *cycle* we always mean a linear combination of the irreducible components E_v of the exceptional divisor with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$.
- The *anticanonical cycle* is the unique cycle Z_K (supported on E) numerically equivalent to an anticanonical divisor. It can be identified by the adjuction formulas $(Z_K, E_v) = E^2 - 2g_v + 2$ (in our setup, we always have $g_v = 0$).
- Note that the graph G and the link M determine each other, modulo a small list of operations on the graph (Neumann).

Computation sequences

A computation sequence $\gamma = (Z_i)_{i=0}^k$ is a sequence of cycles satisfying

- $Z_0 = 0$ and $Z_k = Z_K$.
- For all $0 \leq i < k$ there is a $v(i) \in \mathcal{V}$ so that $Z_{i+1} = Z_i + E_{v(i)}$.

Such sequences give topological upper bounds on p_g. We have

$$p_g = \dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_K))}.$$

The long exact sequence associated to

$$0 \to \mathcal{O}_{\tilde{X}}(-Z_{i+1}) \to \mathcal{O}_{\tilde{X}}(-Z_i) \to \mathcal{O}_{E_{v(i)}}((-Z_i, E_{v(i)})) \to 0$$

provides

$$\dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i))}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1}))} \leq \max\{0, (-Z_i, E_v) + 1\}. \qquad (1)$$

Computation sequences

Summing this up gives

$$p_g \leq \sum_{i=0}^{k-1} \max\{0, (-Z_i, E_v) + 1\}$$

with equality if and only if we have equality in 1 for all i. For lattice cohomological reasons we deonte the right hand side above by $\operatorname{eu}\mathbb{H}(\gamma)$.

Theorem

Assume that X is a hypersurface given by an equation $f = 0$ and f has Newton nondegenerate principal part. Assume further that $H_1(M, \mathbb{Q}) = 0$. Then there exists a computation sequence γ on the minimal good resolution graph of X for which $p_g = \operatorname{eu}\mathbb{H}(\gamma)$. Furthermore, this sequence can be obtained directly from the plumbing graph of M.

Notation—the resolution graph

- For $v \in \mathcal{V}$, let δ_v be the number of neighbours to v in G.
- Let $\mathcal{N}$ be the set of nodes, that is, vertices v with $\delta_v \geq 3$.
- Let $\mathcal{E}$ be the set of ends, that is, vertices v with $\delta_v = 1$.

Newton diagrams

- Let $f \in \mathcal{O}_{\mathbb{C}^3,0}$ be given by a powerseries as $f = \sum_\alpha a_\alpha x^\alpha$. Let $\operatorname{supp} f = \{\alpha \in \mathbb{N}^3 \mid a_\alpha \neq 0\}$.
- The *Newton polyhedron* of f is $\Gamma_+(f) = \operatorname{conv}(\operatorname{supp}(f) + \mathbb{R}^3_{\geq 0})$.
- Let $\mathcal{F}$ be the set of faces of the Newton polyhedron and $\mathcal{F}_c$ the set of compact ones. Then $\Gamma(f) = \cup \mathcal{F}_c$ is the *Newton diagram* of f.
- Assuming nondegeneracy, Oka constructed an embedded resolution of $(\{f = 0\}, 0) \subset (\mathbb{C}^3, 0)$ whose graph is "dual" to the Newton diagram. From now on, G is this resolution.
- Braun and Némethi proved that, assuming some weak conditions on the diagram (obtained after an equisingular deformation), G is the minimal good resolution graph.
- There is a bijection $\mathcal{N} \leftrightarrow \mathcal{F}_c$, $n \mapsto F_n$ so that $n, n' \in \mathcal{N}$ are connected in G by a bamboo if and only if $\dim(F_n \cap F_{n'}) = 1$.

Example

On the picture below we see the diagram of
$f = x^4 + x^3y^2 + y^{10} + x^2z^3 + y^3z^4 + z^8$.

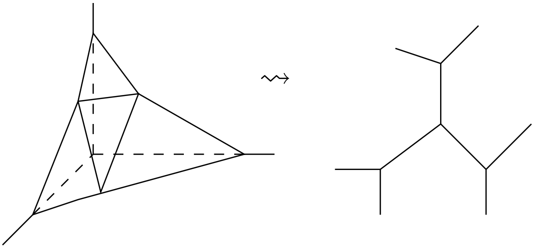

Figure: A Newton diagram and the corresponding resolution graph

Newton diagrams for divisors

- For each $n \in \mathcal{N}$ let ℓ_n be the unique integral primitive functional on $\mathbb{R}^3$ taking constant positive value on F_n. These define weights on the monomials.
- One can assign functionals ℓ_v to all $v \in \mathcal{V}$ so that for all $v \in \mathcal{V}$ we have $E_v^2 \ell_v + \sum_u \ell_u + \ell_{v^*} = 0$ where we sum over neigbours of v. Here
  - $\ell_{v^*} = 0$ if $v \notin \mathcal{E}$.
  - If $v \in \mathcal{E}$ is the end of a bamboo between $F \in \mathcal{F}_c$ and $F' \in \mathcal{F} \setminus \mathcal{F}_c$,then ℓ_{v^*} is the support function of F'.
- Let $\mathcal{V}^e = \mathcal{V} \cup \{v^* | v \in \mathcal{E}\}$. For a cycle $Z = \sum_v m_n(Z) E_v$ let

$$\Gamma_+^e(Z) = \{\alpha \in \mathbb{R}^3 \,|\, \forall v \in \mathcal{V}^e : \ell_v(\alpha) \geq m_v(Z)\}$$

where we set $m_{v^*}(Z) = -1$ for $v \in \mathcal{E}$.

Weights and valuations

Let $g \in \mathcal{O}_{\mathbb{C}^3,0}$ and $\bar{g} \in \mathcal{O}_{X,0}$ its restriction. Let $v \in \mathcal{V}$.

- Let $\mathrm{wt}_v\, g = \min_{p \in \mathrm{supp}\, g} \ell_v(p)$.
- Let $\mathrm{wt}\, g = \sum_v \mathrm{wt}_v(g) E_v$.
- Let $\mathrm{div}_v\, \bar{g}$ be the order of vanishing of the pullback of $\bar{g}$ to $\tilde{X}$.
- Let $\mathrm{div}\, g = \mathrm{div}\, \bar{g} = \sum_v \mathrm{div}_v(g) E_v$.

Oka proved the formula

$$Z_K - E = \mathrm{wt}\, f - \mathrm{wt}(xyz)$$

which yields

$$\Gamma_+(Z_K - E) = \Gamma_+(f) - (1,1,1).$$

A relative Artin cycle

For any $Z \in L$ there exists a $c(Z)$ satisfying

- For $n \in \mathcal{N}$ we have $m_n(c(Z)) = m_n(Z)$.
- For $v \in \mathcal{V} \setminus \mathcal{N}$ we have $(c(Z), E_v) \leq 2 - \delta_v$.
- $c(Z)$ is minimal with respect to the above conditions.

This satisfies the following:

- Monotonicity: If $Z_1 \leq Z_2$ then $c(Z_1) \leq c(Z_2)$.
- Idempotency: $c(c(Z)) = c(Z)$.
- We have $c(Z_K - E) = Z_K - E$ and $c(0) = 0$ (unless our singularity is A_n, but this case is not interesting).
- If $Z \leq c(Z)$, we can compute $c(Z)$ inductively as follows: Take $Z_0 = Z$. Next, assume that we have constructed $Z_0, \ldots, Z_i$ and that $Z_i \neq c(Z)$. Then there is a $v(i)$ so that $(Z_i, E_{v(i)}) > 2 - \delta_{v(i)}$. Define $Z_{i+1} = Z_i + E_{v(i)}$. This sequence ends with $c(Z)$.

The sequence

The sequence is constructed as follows:

- Let $\bar{Z}_0 = 0$.
- Assume we have $\bar{Z}_i$ for some i and that $\bar{Z}_i < Z_K - E$. Choose $\bar{v}(i) \in \mathcal{N}$ so that $m_{\bar{v}(i)}(\bar{Z}_i)/m_{\bar{v}(i)}(Z_K - E)$ is minimal and set $\bar{Z}_{i+1} = c(\bar{Z}_i + E_{v(i)})$
- Using monotonicity and itempotency, one quickly obtains $\bar{Z}_i + E_{v(i)} \leq c(\bar{Z}_i + E_{v(i)})$, which yields a sequence between $\bar{Z}_i$ and $\bar{Z}_{i+1}$.
- These connect together to form a sequence (Z_i) from 0 to $Z_K - E$.
- This suffices as, in fact, $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_K)) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-(Z_K - E)))$.

The plan of the proof

- For all i let $P_i = \mathbb{N}^3 \cap \Gamma_+^e(Z_i) \setminus \Gamma_+^e(Z_{i+1})$. This gives a partition $\mathbb{N}^3 \setminus \Gamma_+^e(Z_K - E) = \coprod_i P_i$.
- We want equality in the inequality

$$\dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i))}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1}))} \leq \max\{0, (-Z_i, E_v) + 1\}.$$

- This is obtained by proving

$$\max\{0, (-Z_i, E_v) + 1\} \leq |P_i| \leq \dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i))}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1}))}.$$

- In particular, we recover the well known formula

$$p_g = |\mathbb{N}^3 \setminus \Gamma_+^e(Z_K - E)| = |\mathbb{Z}_{>0}^3 \setminus \Gamma_+(f)|.$$

What is this P_i?

- For $n \in \mathcal{N}$, let $F_n(Z_K - E) = \Gamma_n(Z_K - E) \cap \{\ell_n = m_n(Z_K - E)\}$, that is, the face of $\Gamma_n(Z_K - E)$ corresponding to n and $C_n = \mathbb{R}_{\geq 0} F_n$.
- From the construction of the sequence Z_i one proves

  $$P_i = C_{v(i)} \cap \{\ell_{v(i)} = m_{v(i)}(Z_i)\} \cap \mathbb{Z}^3.$$

  A simple manipulation shows that

  $$P_i = \{p \in \mathbb{Z}^3 \mid \ell_{v(i)} = m_{v(i)}(Z_i), \forall u \in \mathcal{V}_{v(i)} : \ell_u(p) \geq m_u(Z_i)\}.$$

- By Oka's construction, if u is a neighbour of $v(i)$, then ℓ_u restricts to a *primitive* function on the hyperplane $\{\ell_{v(i)} = m\}$.
- (In fact, complications arise for integral points in the intersection of two cones, but these are technical and tedious and do not cause any serious obstructions)

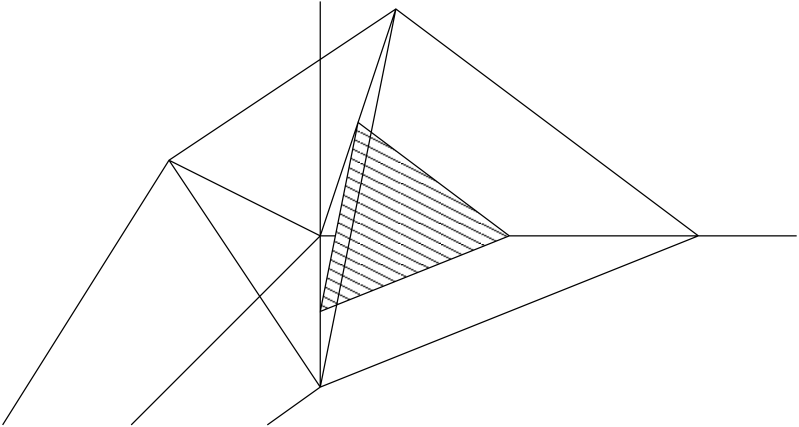

Figure: The set P_i sits inside the triangle shown

Polygons and intersection numbers

The following lemma only holds for very special polygons F, but luckily, these are exactly the ones that show up in our calculations.

Lemma

Let A be a two dimensional affine space with a lattice L and $F \subset A$ a polygon given by integral primitive affine functions $\ell_j : A \to \mathbb{R}$ and values $-1 < r_j \leq 0$. That is, $F = \{a \in A \,|\, \ell_j(a) \geq r_j\}$. Then the function $\sum_j \ell_j$ has constant value c satisfying $|F \cap L| = \max\{0, c+1\}$.

This lemma is applied to the case of $A = \{\ell_{v(i)} = m_{v(i)}(Z_i)\}$ and the restrictions $\ell_u|_A$ for neighbours u of $v(i)$. More precisely, take $p \in A$ and write $Z_i = \sum m_v E_v$.

$$\begin{aligned}(-Z_i, E_{v(i)}) &= -\,E^2_{v(i)} m_{v(i)} - \sum_u m_u \\ &= -\,E^2_{v(i)} \ell_{v(i)}(p) - \sum_u m_u = \sum_u \ell_u(p) - m_u.\end{aligned}$$

From the lemma we now get $|P_i| = \max\{0, (-Z_i, E_{v(i)}) + 1\}$.

The second inequality

For the last inequality it is enough to prove

- If $\alpha \in P_i$ then $x^\alpha \in H^0(\tilde{X},\ O_{\tilde{X}}(-Z_i))$.
- The family $(x^\alpha)_{\alpha \in P}$ is linearly independent modulo $H^0(\tilde{X},\ O_{\tilde{X}}(-Z_{i+1}))$

The first item is clear since $\operatorname{div} x^\alpha = \operatorname{wt} x^\alpha$. For the second one, take coefficients a_α for $\alpha \in P_i$ (not all zero) and set $g = \sum_{\alpha \in P_i} a_\alpha x^\alpha$. One proves easily that the set P_i is contained in a segment, and that this does not hold for $\operatorname{supp} f_{v(i)}$, where $f_{v(i)}$ is the principal part of f w.r.t. the weight function $\ell_{v(i)}$. In particular, $f_{v(i)}$ does not divide g, even over the ring $\mathcal{O}_{X,0}[x^{-1}, y^{-1}, z^{-1}]$. By a lemma of Ebeling and Gusein-Zade, this means that $\operatorname{div}_{v(i)} g = \operatorname{wt}_{v(i)} g$, hence $g \notin H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1}))$. This proves the second item.