Chapter 4, Neuro-Fuzzy and Soft Computing: Fuzzy Inference Systems
Outline

• Introduction (4.1)
• Mamdani Fuzzy models (4.2)
• Sugeno Fuzzy Models (4.3)
• Tsukamoto Fuzzy models (4.4)
• Other Considerations (4.5)
  • Fuzzy modeling
Fuzzy inference is a computer paradigm based on fuzzy set theory, fuzzy if-then-rules and fuzzy reasoning.

Applications: data classification, decision analysis, expert systems, times series predictions, robotics & pattern recognition.

Different names; fuzzy rule-based system, fuzzy model, fuzzy associative memory, fuzzy logic controller & fuzzy system.
Introduction (4.1) (cont.)

Structure

- **Rule base** ← selects the set of fuzzy rules
- **Database (or dictionary)** ← defines the membership functions used in the fuzzy rules
- **A reasoning mechanism** ← performs the inference procedure (derive a conclusion from facts & rules!)

**Defuzzification:** extraction of a crisp value that best represents a fuzzy set

- **Need:** it is necessary to have a crisp output in some situations where an inference system is used as a controller
Fuzzy If-Then Rules

• **Mamdani style**
  If pressure is high then volume is small

• **Sugeno style**
  If speed is medium then resistance = 5*speed
Fuzzy Inference System (FIS)

If speed is low then resistance = 2
If speed is medium then resistance = 4*speed
If speed is high then resistance = 8*speed

Rule 1: \( w_1 = .3; r_1 = 2 \)
Rule 2: \( w_2 = .8; r_2 = 4*2 \)
Rule 3: \( w_3 = .1; r_3 = 8*2 \)

Resistance = \( \frac{\sum (w_i*r_i)}{\sum w_i} \) = 7.12
Fuzzy Inference System (FIS)

Block diagram for a fuzzy inference system
Non linearity

- In the case of crisp inputs & outputs, a fuzzy inference system implements a nonlinear mapping from its input space to output space.
Mamdani Fuzzy models [1975] (4.2)

Goal: Control a steam engine & boiler combination by a set of linguistic control rules obtained from experienced human operators.

Illustration of how a two-rule Mamdani fuzzy inference system derives the overall output $z$ when subjected to two crisp inputs $x$ & $y$. 
Mamdani FIS

A1, B1, C1

A2, B2, C2

X, Y, Z

T-norm, Max

w1, w2
Defuzzification [definition]

“It refers to the way a crisp value is extracted from a fuzzy set as a representative value”

- There are five methods of defuzzifying a fuzzy set $A$ of a universe of discourse $Z$
  - Centroid of area $z_{COA}$
  - Bisector of area $z_{BOA}$
  - Mean of maximum $z_{MOM}$
  - Smallest of maximum $z_{SOM}$
  - Largest of maximum $z_{LOM}$
Mamdani Fuzzy models (4.2) (cont.)

- Centroid of area $z_{COA}$
  
  $z_{COA} = \frac{\int z \mu_A(z) dz}{\int \mu_A(z) dz},$

  where $\mu_A(z)$ is the aggregated output MF.

- Bisector of area $z_{BOA}$
  
  this operator satisfies the following;

  $\int_{\alpha}^{\beta} \mu_A(z) dz = \int_{z_{BOA}}^{\mu_A(z)} dz,$

  where $\alpha = \min \{z; z \in Z\} \& \beta = \max \{z; z \in Z\}$. The vertical line $z = z_{BOA}$ partitions the region between $z = \alpha$, $z = \beta$, $y = 0$ & $y = \mu_A(z)$ into two regions with the same area.
- Mean of maximum $z_{\text{MOM}}$

This operator computes the average of the maximizing $z$ at which the MF reaches a maximum $\mu$. It is expressed by:

$$z_{\text{MOM}} = \frac{\int z \, dz}{\int_0^{z'} \, dz}$$

where $Z' = \{z; \mu_A(z) = \mu\}$

By definition: if $\mu_A(z)$ has a single maximum at $z = z^*$ then $z_{\text{MOM}} = z^*$

However: if $\max_z \mu_A(z) = [z_1, z_2]$ then $z_{\text{MOM}} = \frac{z_1 + z_2}{2}$
Mamdani Fuzzy models (4.2) (cont.)

- Smallest of maximum $z_{SOM}$
  
  Amongst all $z$ that belong to $[z_1, z_2]$, the smallest is called $z_{SOM}$

- Largest of maximum $z_{LOM}$
  
  Amongst all $z$ that belong to $[z_1, z_2]$, the largest value is called $z_{LOM}$
Various defuzzification schemes for obtaining a crisp output
Defuzzification: Example

Rule 1: IF $X$ is $A_1$ and $Y$ is $B_1$
    THEN $Z$ is $C_1$

Rule 2: IF $X$ is $A_2$ and $Y$ is $B_2$
    THEN $Z$ is $C_2$

$x_0$ and $y_0$ are the sensor readings for fuzzy variables $X$ and $Y$
Example (Cont..)

Membership functions:

\[ \mu_{A_1} = \begin{cases} \frac{x-2}{3} & 2 \leq x \leq 5 \\ \frac{5-x}{3} & 5 < x \leq 8 \end{cases} \]

\[ \mu_{B_1} = \begin{cases} 1-x & 5 \leq y \leq 8 \\ 8 < y \leq 11 \end{cases} \]

\[ \mu_{C_1} = \begin{cases} \frac{1-x}{3} & 1 \leq z \leq 4 \\ \frac{4-z}{3} & 4 < z \leq 7 \end{cases} \]

\[ \mu_{A_2} = \begin{cases} \frac{x-3}{3} & 3 \leq x \leq 6 \\ \frac{6-x}{3} & 6 < x \leq 9 \end{cases} \]

\[ \mu_{B_2} = \begin{cases} 1-x & 4 \leq y \leq 7 \\ 7 < y \leq 10 \end{cases} \]

\[ \mu_{C_2} = \begin{cases} \frac{1-y}{3} & 3 \leq z \leq 6 \\ \frac{6-z}{3} & 6 < z \leq 9 \end{cases} \]

Assume sensor values \( x_0 = 4 \) and \( y_0 = 8 \), then calculate

1. the membership function for the control action recommended by the combination of these two rules

2. the crisp value of the control action using the COA and MOM methods.
Defuzzification: Example (cont.)

- For rule 1: \( \mu_{A_1}(x_0) = \frac{2}{3} \) and \( \mu_{B_1}(y_0) = 1 \)
- For rule 2: \( \mu_{A_2}(x_0) = \frac{1}{3} \) and \( \mu_{B_2}(y_0) = \frac{2}{3} \)
- \[
\alpha_1 = \text{Min}(\mu_{A_1}(x_0), \mu_{B_1}(y_0)) = \text{Min}(\frac{2}{3}, 1) = \frac{2}{3}.
\]
- \[
\alpha_2 = \text{Min}(\mu_{A_2}(x_0), \mu_{B_2}(y_0)) = \text{Min}(\frac{1}{3}, \frac{2}{3}) = \frac{1}{3}.
\]
- \[
Z^*_{COA} = \frac{\frac{2}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} + \ldots}{\frac{1}{3} + \frac{2}{3} + \ldots} = 4.7.
\]
- \[
Z^*_{MOM} = \frac{3 + 4 + 5}{3} = 4.0.
\]
Example #1

Single input single output Mamdani fuzzy model with 3 rules:

If X is small then Y is small → R₁
If X is medium then Y is medium → R₂
Is X is large then Y is large → R₃

X = input ∈ [-10, 10]
Y = output ∈ [0, 10]

Using max-min composition (R₁ o R₂ o R₃) and centroid defuzzification, we obtain the following overall input-output curve.
Neuro-Fuzzy and Soft Computing: Fuzzy Sets

Single input (top) single output (bottom) antecedent & consequent MFs
Overall input-output curve
Example #2

Two input single-output Mamdani fuzzy model with 4 rules:

If X is small & Y is small then Z is negative large
If X is small & Y is large then Z is negative small
If X is large & Y is small then Z is positive small
If X is large & Y is large then Z is positive large

X = [-5, 5]; Y = [-5, 5]; Z = [-5, 5] with max-min composition & centroid defuzzification, we can determine the overall input-output surface
Two-input single output antecedent & consequent MFs
Overall input-output surface
Mamdani Fuzzy models (4.2)(cont.)

Other Variants:

• Classical fuzzy reasoning is “not” tractable, difficult to compute

• In practice, a fuzzy inference system may have a certain reasoning mechanism that does not follow the strict definition of the compositional rule of inference
Mamdani Fuzzy models (4.2) (cont.)

- **Reminder:**

\[
C' = (A' \times B') \circ (A \times B \rightarrow C)
\]

\[
\mu_{C'}(z) = \bigvee_{x,y} \left[ \mu_{A'}(x) \land \mu_{B'}(y) \right] \land \left[ \mu_{A}(x) \land \mu_{B}(y) \land \mu_{C}(z) \right]
\]

\[
= \bigvee_{x,y} \left\{ \mu_{A'}(x) \land \mu_{B'}(y) \land \mu_{A}(x) \land \mu_{B}(y) \right\} \land \mu_{C}(z)
\]

\[
= \left\{ \bigvee_{x} \mu_{A'}(x) \land \mu_{A}(x) \right\} \land \left\{ \bigvee_{y} \mu_{B'}(y) \land \mu_{B}(y) \right\} \land \mu_{C}(z)
\]

\[
= (w_1 \land \mu_{C}(z))
\]
- $w_1$ = degree of compatibility between $A$ & $A'$
- $w_2$ = degree of compatibility between $B$ & $B'$
- $w_1 \land w_2$ = degree of fulfillment of the fuzzy rule (antecedent part) = firing strength

- Qualified (induced) consequent MFs represent how the firing strength gets propagated & used in a fuzzy implication statement

- Overall output MF aggregate all the qualified consequent MFs to obtain an overall output MF
Mamdani Fuzzy models (4.2) (cont.)

- One might use **product** for firing strength computation
- One might use **min** for qualified consequent MFs computation
- One might use **max** for MFs aggregation into an overall output MF
Mamdani Fuzzy models (4.2) (cont.)

To completely specify the operation of a Mamdani fuzzy inference system, we need to assign a function for each of the following operators:

- **AND operator** (usually T-norm) for the rule firing strength computation with AND’ed antecedents
- **OR operator** (usually T-conorm) for calculating the firing strength of a rule with OR’ed antecedents
- **Implication operator** (usually T-norm) for calculating qualified consequent MFs based on given firing strength
- **Aggregate operator** (usually T-conorm) for aggregating qualified consequent MFs to generate an overall output MF ≠ composition of facts & rules to derive a consequent
- **Defuzzification operator** for transforming an output MF to a crisp single output value
Example:

⇒ “product” ⊕ “sum”
Aggregate

This sum-product composition provides the following theorem:
Final crisp output when using centroid defuzzification = weighted average of centroids of consequent MFs where:

\[ w \text{ (rule}_i\text{)} = (\text{firing strength})_i \times \text{Area (consequent MFs)} \]

\[ \mu_{C'}(z) = w_1 \mu_{C_1}(z) + w_2 \mu_{C_2}(z) \]

Proof: Use the following:
and compute: \( z_{\text{COA}} \) (centroid defuzzification)

Conclusion: Final crisp output can be computed if:

– Area of each consequent MF is known
– Centroid of each consequent MF is known
Goal: Generation of fuzzy rules from a given input-output data set

A TSK fuzzy rule is of the form:

“If $x$ is $A$ & $y$ is $B$ then $z = f(x, y)$”

Where $A$ & $B$ are fuzzy sets in the antecedent, while $z = f(x, y)$ is a crisp function in the consequent.

$f(.,.)$ is very often a polynomial function w.r.t. $x$ & $y$. 

[Takagi, Sugeno & Kang, 1985]
If $f(.,.)$ is a first order polynomial, then the resulting fuzzy inference is called a **first order Sugeno fuzzy model**

If $f(.,.)$ is a constant then it is a **zero order Sugeno fuzzy model** (special case of Mamdani model)

Case of two rules with a first order Sugeno fuzzy model
- Each rule has a crisp output
- Overall output is obtained via **weighted average**
- No defuzzification required
**First Order Sugeno FIS**

- **Rule base**
  
  \[
  \begin{align*}
  &\text{EF } X \text{ er } A_1 \text{ OG } Y \text{ er } B_1 \implies Z = p_1 x + q_1 y + r_1 \\
  &\text{EF } X \text{ er } A_2 \text{ OG } Y \text{ er } B_2 \implies Z = p_2 x + q_2 y + r_2
  \end{align*}
  \]

- **Fuzzy Reasoning**

\[
\begin{align*}
  w_1 &= p_1 x + q_1 y + r_1 \\
  w_2 &= p_2 x + q_2 y + r_2 \\
  z &= \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}
\end{align*}
\]
Example 1: Single output-input Sugeno fuzzy model with three rules

- If $X$ is small then $Y = 0.1X + 6.4$
- If $X$ is medium then $Y = -0.5X + 4$
- If $X$ is large then $Y = X - 2$

If “small”, “medium” & “large” are nonfuzzy sets then the overall input-output curve is a piece wise linear.
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(a) Antecedent MFs for Crisp Rules

(b) Overall I/O Curve for Crisp Rules
However, if we have smooth membership functions (fuzzy rules) the overall input-output curve becomes a smoother one.
Neuro-Fuzzy and Soft Computing: Fuzzy Sets
**Example 2**: Two-input single output fuzzy model with 4 rules

- $R_1$: if $X$ is small & $Y$ is small then $z = -x + y + 1$
- $R_2$: if $X$ is small & $Y$ is large then $z = -y + 3$
- $R_3$: if $X$ is large & $Y$ is small then $z = -x + 3$
- $R_4$: if $X$ is large & $Y$ is large then $z = x + y + 2$

$$R_1 \rightarrow (x \land s) \& (y \land s) \rightarrow w_1$$
$$R_2 \rightarrow (x \land s) \& (y \land l) \rightarrow w_2$$
$$R_3 \rightarrow (x \land l) \& (y \land s) \rightarrow w_3$$
$$R_4 \rightarrow (x \land l) \& (y \land l) \rightarrow w_4$$

**Aggregated consequent** $\rightarrow F[(w_1, z_1); (w_2, z_2); (w_3, z_3); (w_4, z_4)]$

$= \text{weighted average}$
Antecedent & consequent MFs
Overall input-output surface
Tsukamoto Fuzzy models (4.4) [1979]

It is characterized by the following:

The consequent of each fuzzy if-then-rule is represented by a fuzzy set with a monotonical MF.

The inferred output of each rule is a crisp value induced by the rule’s firing strength.
Tsukamoto Fuzzy models

The Tsukamoto fuzzy model
Example: single-input Tsukamoto fuzzy model with 3 rules

if X is small then Y is $C_1$
if X is medium then Y is $C_2$
if X is large then Y is $C_3$
Antecedent & consequent MFs, each rule’s output, and overall input-output curve
Input Space Partitioning

- The antecedent of a fuzzy rule defines a local fuzzy region such as \((\text{very tall} \times \text{heavy}) \subset (\text{height} \times \text{weight})\)

- The consequent describes the local behavior within the fuzzy region

- There are 3 partitionings
  - Grid partition
  - Tree partition
  - Scatter partition
Input Space Partitioning

- Grid partitioning
- Tree partitioning
- Scatter partitioning

- CART method
- C-means clustering
Other Considerations (4.5) (cont.)

- **Grid partition**
  Each region is included in a square area ⇒ hypercube
  Difficult to partition the input using the Grid in the case of a large number of inputs. If we have k inputs & m MFs for each ⇒ $m^k$ rules!!

- **Tree partition**
  Each region can be uniquely specified along a corresponding decision tree. No exponential increase in the number of rules

- **Scatter partition**
  Each region is determined by covering a subset of the whole input space that characterizes a region of possible occurrence of the input vectors
• We have covered several types of fuzzy inference systems (FIS’s)

• A design of a fuzzy inference system is based on the past known behavior of a target system

• A developed FIS should reproduce the behavior of the target system
Fuzzy Modeling

• Examples of FIS’s

- Replace the human operator that regulates & controls a chemical reaction, a FIS is a fuzzy logic controller

- Target system is a medical doctor; a FIS becomes a fuzzy expert system for medical diagnosis
How to construct a FIS for a specific application?

- Incorporate human expertise about the target system: it is called the domain knowledge (linguistic data!)

- Use conventional system identification techniques for fuzzy modeling when input-output data of a target system are available (numerical data)
General guidelines about fuzzy modeling

A. Identification of the surface structure

i. Select relevant input-output variables
ii. Choose a specific type of FIS
iii. Determine the number of linguistic terms associated with each input & output variables (for a Sugeno model, determine the order of consequent equations)

Part A describes the behavior of the target system by means of linguistic terms
B. Identification of deep structure

i. Choose an appropriate family of parameterized MFs

ii. Interview human experts familiar with the target systems to determine the parameters of the MFs used in the rule base

iii. Refine the parameters of the MFs using regression & optimization techniques (best performance for a plant in control!)

(i) + (ii): assumes the availability of human experts

(iii): assumes the availability of the desired input-output data set
• Applications

- Design a digit recognizer based on a FIS. View each digit as a matrix of 7*5 pixels

- Design a character recognizer based on a FIS. View each character as a matrix of 7*5 pixels