Disc formulas for $\omega$-plurisubharmonic functions

Benedikt Steinar Magnússon

Science Institute
University of Iceland
http://www.hi.is/~bsm

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The Poisson disc functional

\[ \sup\{u(x); u \in PSH(X), u \leq \varphi\} = \inf \left\{ \int_\mathbb{T} \varphi \circ f \, d\sigma; f \in A_X, f(0) = x \right\} \]

where \( A_X \) is the set of all closed holomorphic discs in \( X \) and \( \sigma \) is the arc length measure on the unit circle \( \mathbb{T} \) normalized to 1.

▶ Poletsky [1993]: \( \varphi \) usc and \( X \) a domain in \( \mathbb{C}^n \)
▶ Lárusson/Sigurdsson [1998,2003] and Rosay [2003]: \( \varphi \) usc and \( X \) any complex manifold
▶ Edigarian [2003]: \( \varphi \) plurisuperharmonic
▶ Drinovec Drnovšek/Forstnerič [2011]: \( \varphi \) usc, \( X \) irreducible complex space
▶ M [2011]: \( \varphi = \varphi_1 - \varphi_2 \), where \( \varphi_1 \) usc and \( \varphi_2 \) psh, \( X \) complex manifold

QUESTION: Is there a similar result for \( \omega \)-psh (quasi-psh) functions?
The Poisson disc functional

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\sup\{u(x); u \in PSH(X), u \leq \varphi\} = \inf \left\{ \int_{\mathbb{T}} \varphi \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x \right\}
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where \( \mathcal{A}_X \) is the set of all closed holomorphic discs in \( X \) and \( \sigma \) is the arc length measure on the unit circle \( \mathbb{T} \) normalized to 1.

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where $\mathcal{A}_X$ is the set of all closed holomorphic discs in $X$ and $\sigma$ is the arc length measure on the unit circle $\mathbb{T}$ normalized to 1.

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**QUESTION:** Is there a similar result for $\omega$-psh (quasi-psh) functions?
$\omega$-settings

- $X$ a complex manifold
- $\omega = \omega_1 - \omega_2$ the difference of two positive closed $(1, 1)$-currents on $X$
- then we have local potentials $\psi = \psi_1 - \psi_2$ such that $dd^c \psi = \omega$, and $\psi_1$ and $\psi_2$ are psh
- let $\text{sing}(\omega) = \bigcup (\psi_1^{-1}(-\infty) \cup \psi_2^{-1}(-\infty))$
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**ω-psh functions**

**Definition:** A function $u$ is $\omega$-usc if $u + \psi$ is usc for every local potential $\psi$ of $\omega$ and $\limsup_{y \to x} u(y) = u(x)$ for $x \in \text{sing}(\omega)$.
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**Definition:** An $\omega$-usc function $u$ is $\omega$-psh if $dd^c u \geq -\omega$, that is $u + \psi$ is psh for every local potential $\psi$ of $\omega$.

Let $\mathcal{PSH}(X, \omega)$ denote the set of $\omega$-psh functions on $X$. 
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**Remark:** Whenever the sum of two function is not defined (one is $+\infty$ and the other $-\infty$) we use $\limsup$. 
Connecting $\omega$ and $A_X$

If $f \in A_X$ we define the pullback of $\omega$ by $f$, denoted $f^*\omega$, locally by $\Delta(\psi \circ f)$. 
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Let $R_{f^*\omega}(t) = \int_D \log \left| \frac{t-s}{1-ts} \right| f^*\omega(s)$ be its Riesz potential.
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**Proposition:** For an $\omega$-usc function $u$ the following is equivalent

- $u \in \mathcal{P}SH(X, \omega)$
- $u \circ f$ is in $\mathcal{S}H(\mathbb{D}, f^*\omega)$ for all $f \in \mathcal{A}_X$
- $u \circ f + R_{f^*\omega}$ is subharmonic for all $f \in \mathcal{A}_X$
Connecting $\omega$ and $A_X$ cont.

Assume $\varphi = \varphi_1 - \varphi_2$, where $\varphi_1$ is $\omega_1$-usc and $\varphi_2$ is psh (in particular $\varphi$ can be usc).
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If $u \in PSH(X, \omega)$, $u \leq \varphi$ and $f \in A_X$, $f(0) = x$ then

$$u(f(0)) + R_{f^*\omega}(0) \leq \int_T u \circ f \, d\sigma + \int_T R_{f^*\omega} \, d\sigma$$
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Taking supremum over $u$, and infimum over $f$ we get

$$\sup \{ u(x); u \in PSH(X, \omega), u \leq \varphi \} \leq \inf \{ -R_{f^*}\omega(0) + \int_T \varphi \circ f \, d\sigma; f \in A_X, f(0) = x \}.$$
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$$u(f(0)) + R_{f^*}\omega(0) \leq \int_\mathbb{T} u \circ f \, d\sigma + \int_\mathbb{T} R_{f^*}\omega \, d\sigma$$

that is

$$u(x) \leq -R_{f^*}\omega(0) + \int_\mathbb{T} \varphi \circ f \, d\sigma.$$
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**THEOREM:** Let $X$ be a complex manifold, $\omega = \omega_1 - \omega_2$ be the difference of two closed positive $(1, 1)$-currents on $X$, $\varphi = \varphi_1 - \varphi_2$ be the difference of an $\omega_1$-upper semicontinuous function $\varphi_1$ and a plurisubharmonic function $\varphi_2$. Then the function $\sup\{ u \in \mathcal{P}SH(X, \omega); u \leq \varphi \}$ is $\omega$-plurisubharmonic and for every $x \in X \setminus \text{sing}(\omega)$,

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\sup\{ u(x); u \in \mathcal{P}SH(X, \omega), u \leq \varphi \} = \inf\{ -R_f^\ast \omega(0) + \int_T \varphi \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x \}.
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Main theorem

**Theorem:** Let $X$ be a complex manifold, $\omega = \omega_1 - \omega_2$ be the difference of two closed positive $(1, 1)$-currents on $X$, $\varphi = \varphi_1 - \varphi_2$ be the difference of an $\omega_1$-upper semicontinuous function $\varphi_1$ and a plurisubharmonic function $\varphi_2$. Then the function $\sup\{u \in PSH(X, \omega); u \leq \varphi\}$ is $\omega$-plurisubharmonic and for every $x \in X \setminus \text{sing}(\omega)$,

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Back to the classical case - Corollary

RIESZ DISC FUNCTIONAL: For a given psh function $v$ it has been shown that

$$\sup\{u(x); u \in \mathcal{PSH}(X), dd^c u \geq dd^c v, u \leq 0\}$$

$$= \inf\left\{\frac{1}{2\pi} \int_{\mathbb{D}} \log |\cdot| \Delta(v \circ f); f \in \mathcal{A}_X, f(0) = x\right\}.$$
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Letting $\omega = -dd^c v$ in our main theorem we can combine this with the Poisson disc functional,

$$\sup \{ u(x); u \in \mathcal{PSH}(X), dd^c u \geq dd^c v, u \leq \phi \} = \inf \left\{ \frac{1}{2\pi} \int_{\mathbb{D}} \log |\cdot| \Delta(v \circ f) + \int_{\mathbb{T}} \phi \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x \right\},$$

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$$\sup\{u(x); u \in \mathcal{P}SH(X, -dd^c \nu), dd^c u \geq dd^c \nu, u \leq \varphi\}$$

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where \( \varphi = \varphi_1 - \varphi_2 \) is the difference of usc function \( \varphi_1 \) and a psh function \( \varphi_2 \).
Proof

We know that

\[ \sup\{u(x); u \in PSH(X, \omega), u \leq \varphi\} \]

\[ \leq \inf\{-R_{f^*}\omega(0) + \int_T \varphi \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\} =: \hat{\varphi}, \]
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so if $\hat{\varphi} \leq \varphi$ and $\hat{\varphi}$ is $\omega$-psh, then it is in the family on the left hand side and we have an equality.
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Now if \( f_x \) is the constant disc which maps everything to \( x \in X \), then by the definition of the envelope

\[ \hat{\varphi}(x) \leq -R_{f_x^*\omega}(0) + \int_T \varphi \circ f_x \, d\sigma = \varphi(x) \]
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$$\hat{\varphi}(x) \leq -R_{f_x^*}(0) + \int_T \varphi \circ f_x \, d\sigma = \varphi(x)$$

The hard part is to show that the $\hat{\varphi}$ is $\omega$-psh
Proof in the case of a global potential

Assume there is a function $\psi = \psi_1 - \psi_2$ on $X$ such that $dd^c\psi = \omega$.
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Assume there is a function $\psi = \psi_1 - \psi_2$ on $X$ such that $dd^c \psi = \omega$. Then for $f \in A_X$, $f(0) = x$, by the Riesz rep. formula

$$\psi(f(0)) = R_{f^* \omega}(0) + \int_T \psi \circ f \, d\sigma,$$
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$$\psi(f(0)) = R_{f^*} \omega(0) + \int_{\mathbb{T}} \psi \circ f \, d\sigma,$$

and then we can show that $\hat{\varphi}$ is $\omega$-psh because

$$\hat{\varphi}(x) + \psi(x) = \inf \left\{ -R_{f^*} \omega(0) + \int_{\mathbb{T}} \varphi \circ f \, d\sigma ; f \in \mathcal{A}_X, f(0) = x \right\} + \psi(x)$$
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$$= \inf \left\{ -R_{f^*}\omega(0) + \int_{\mathcal{T}} \phi \circ f \, d\sigma + R_{f^*}\omega(0) + \int_{\mathcal{T}} \psi \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x \right\}$$
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$$= \inf \{ \int_T (\varphi + \psi) \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\}$$
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$$= \inf\{ \int_{\mathbb{T}} (\varphi + \psi) \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\}$$

This function is psh since $\varphi + \psi = (\varphi + \psi_1) - \psi_2$ is the difference of an usc function and a plurisubharmonic function.
Proof in the general case

We can prove that the envelope \( \hat{\varphi} \) is \( \omega \)-psh by showing that it satisfies a sub-average property for analytic discs.

\[
(\hat{\varphi} + \psi)(f(0)) \leq \int_{\mathbb{T}} (\hat{\varphi} + \psi) \circ f \, d\sigma.
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That is, for $\varepsilon > 0$ there exists a disc $g$ such that $g(0) = f(0)$ and

$$(\hat{\phi} + \psi)(f(0)) \leq -R_{g^*\omega}(0) + \int_{\mathcal{T}} \phi \circ g \, d\sigma \leq \int_{\mathcal{T}} (\hat{\phi} + \psi) \circ f \, d\sigma + \varepsilon.$$
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To find $g$ we embedd a ”big enough” part of $X$ into $X \times \mathbb{C}^2$ and show that there we have a global potential.
Proof in the general case

We can prove that the envelope \( \hat{\varphi} \) is \( \omega \)-psh by showing that it satisfies a sub-average property for analytic discs.

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(\hat{\varphi} + \psi)(f(0)) \leq \int_{T} (\hat{\varphi} + \psi) \circ f \, d\sigma.
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That is, for \( \varepsilon > 0 \) there exists a disc \( g \) such that \( g(0) = f(0) \) and

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(\hat{\varphi} + \psi)(f(0)) \leq -R_g^* \omega(0) + \int_{T} \varphi \circ g \, d\sigma \leq \int_{T} (\hat{\varphi} + \psi) \circ f \, d\sigma + \varepsilon.
\]

To find \( g \) we embedd a "big enough" part of \( X \) into \( X \times \mathbb{C}^2 \) and show that there we have a global potential.

Since we have a global potential on this subset in \( X \times \mathbb{C}^2 \) the corresponding envelope there is \( \omega \)-psh and we have a "good disc" \( \tilde{g} \) in \( X \times \mathbb{C}^2 \). The disc \( g = \pi \circ \tilde{g} \) is then the disc we are looking for.
\( \omega \)-Reduction theorem

Let \( X \) be a complex manifold, \( H \) a disc functional on \( A_X \) and \( \omega = \omega_1 - \omega_2 \) the difference of two positive, closed \((1, 1)\)-currents on \( X \). The envelope \( EH \) is \( \omega \)-plurisubharmonic if it satisfies the following.

(i) \( E\Phi^*H \) is \( \Phi^*\omega \)-plurisubharmonic for every holomorphic submersion \( \Phi \) from a complex manifold where \( \Phi^*\omega \) has a global potential.

(ii) There is an open cover of \( X \) by subsets \( U \), with \( \omega \)-pluripolar subsets \( Z \subset U \) and local potentials \( \psi \) on \( U \), \( \psi^{-1}(\{-\infty\}) \subset Z \), such that for every \( h \in A_U \) with \( h(\overline{D}) \not\subset Z \), the function \( t \mapsto (H(h(t)) + \psi(h(t)))^{\dagger} \) is dominated by an integrable function on \( \mathbb{T} \).

(iii) If \( h \in A_X \), \( h(0) \not\in \text{sing}(\omega) \), \( t_0 \in \mathbb{T} \setminus h^{-1}(\text{sing}(\omega)) \) and \( \varepsilon > 0 \), then \( t_0 \) has a neighbourhood \( U \) in \( \mathbb{C} \) and there is a local potential \( \psi \) in a neighbourhood of \( h(U) \) such that for all sufficiently small arcs \( J \) in \( \mathbb{T} \) containing \( t_0 \) there is a holomorphic map \( F : D_r \times U \to X \), \( r > 1 \), such that \( F(0, \cdot) = h|_U \) and