The Ackermann Function

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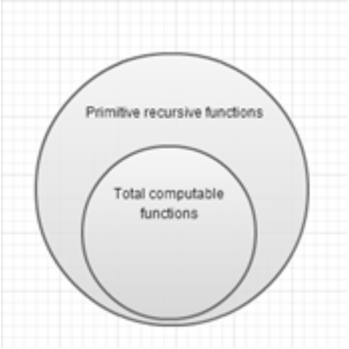
What is it?

- One of the simplest and earliest-discovered examples of a total function that is not pri. recursive.



- In the early 1900s it was believed that every computable function was also pri. recursive.

Big Numbers

- Googol: 10^100
- GoogolPlex: 10^10^100
- The Ackermann function grows so that it’s output becomes larger than a GoogolPlex rather quickly.
- Graham number

The function itself

- The original Ackermann function had three non-negative arguments.
    - Provided by Wilhelm Ackermann.
- The most common version is generally known as the two-argument Ackermann-Péter function.

The function itself

- Defined as follows for non-negative integers m and n, and is recursively defined:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

Evaluation

$$
\begin{aligned}
A(1,2) &= A(0, A(1,1)) \\
&= A(0, A(0, A(1,0))) \\
&= A(0, A(0, A(0,1))) \\
&= A(0, A(0,2)) \\
&= A(0,3) \\
&= 4.
\end{aligned}
$$

Implementation

```php
<?php
    function A($m, $n)
    {
        if($m == 0)
        {
            return $n + 1;
        }
        else if($m > 0 && $n == 0)
        {
            return A($m - 1, 1);
        }
        else if($m > 0 && $n > 0)
        {
            return A($m - 1, A($m, $n - 1));
        }
    }
    echo "<p>A(0, 4): ". A(0, 4). "</p>";
    echo "<p>A(4, 0): ". A(4, 0). "</p>";
    echo "<p>A(3, 2): ". A(3, 2). "</p>";
?>
```

A(0, 4): 5

A(4, 0): 13

A(3, 2): 29

```php
echo "<p>A(4, 2): </p>";
echo A(4, 2);
```

A(4, 2):

Fatal error: Maximum execution time of 300 seconds exceeded

About the function

- The evaluation of A(m, n) always terminates.

- It‘s recursion is bounded because in each recursive application either m decreases, or m remains the same and n decreases. Each time that n reaches zero, m decreases, so m eventually will reach zero as-well.

- However, when m decreases **there is no upper bound on how much n can increase** – and it will often increase greatly.

- There is no alternative presentation of the Ackermann function that uses only primitive recursion so it **cannot be primitive recursive**.

Table of values

m\n	0	1	2	3	4	n
0	1	2	3	4	5	$n+1$
1	2	3	4	5	6	$n+2=2+(n+3)-3$
2	3	5	7	9	11	$2n+3=2\cdot(n+3)-3$
3	5	13	29	61	125	$2^{(n+3)}-3$
4	13 $=2^{2^2}-3$	65533 $=2^{2^{2^2}}-3$	$2^{65536}-3$ $=2^{2^{2^{2^2}}}-3$	$2^{2^{65536}}-3$ $=2^{2^{2^{2^{2^2}}}}-3$	$2^{2^{2^{65536}}}-3$ $=2^{2^{2^{2^{2^{2^2}}}}}-3$	$\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n+3}-3$

Identical function calls

m\n	0	1	2	3	4	n
0	0+1	1+1	2+1	3+1	4+1	$n+1$
1	A(0,1)	A(0,A(1,0))	A(0,A(1,1))	A(0,A(1,2))	A(0,A(1,3))	$n+2=2+(n+3)-3$
2	A(1,1)	A(1,A(2,0))	A(1,A(2,1))	A(1,A(2,2))	A(1,A(2,3))	$2n+3=2\cdot(n+3)-3$
3	A(2,1)	A(2,A(3,0))	A(2,A(3,1))	A(2,A(3,2))	A(2,A(3,3))	$2^{(n+3)}-3$
4	A(3,1)	A(3,A(4,0))	A(3,A(4,1))	A(3,A(4,2))	A(3,A(4,3))	$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n+3}-3$
5	A(4,1)	A(4,A(5,0))	A(4,A(5,1))	A(4,A(5,2))	A(4,A(5,3))	A(4, A(5, n-1))
6	A(5,1)	A(5,A(6,0))	A(5,A(6,1))	A(5,A(6,2))	A(5,A(6,3))	A(5, A(6, n-1))

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http://en.wikipedia.org/wiki/Ackermann_function

https://www.youtube.com/watch?v=CUbDmWIFYzo