TÖL101F Algorithms, Logic and Complexity

Final exam **December 11th, 2008**
Teacher: Hjálmtýr Hafsteinsson **Time: $9^{00} – 12^{00}$**

All problems have the same value. ***You only have to solve 5 problems out of 6. The five best problems count.*** *All written material and a calculator are allowed.*

- Note that an answer without justification is worthless. Therefore justify all answers and remember that it is not necessary to write up definitions that are in the book.
- You can answer in either English or Icelandic.

1. For each of the following statements say whether it is true or false. Give a short proof (or a counterexample) in each case.

a) If A is not a regular language, then the complement of A can not be regular.

b) 2SAT $\leq_p$ 3SAT.

c) If A ∩ B is NP-complete, A ∈ NP, and B ∈ P, then A must be NP-complete.

d) In primality testing, Fermat's Little Theorem implies that if *c* is a composite number then it will not satisfy the equation $a^{c-1} \equiv 1 \pmod{c}$ for any *a* in the set {1, … *c*-1}.

2. Let $A = \{$ "x/y" | x and y are positive decimal integers with $x < y$ $\}$. The set *A* contains strings that represent all possible decimal fractions (i.e. rational numbers less than 1). Thus the strings "23/48" and "7/282" are in *A*, but "49/23" is not. Use the pumping lemma to prove that *A* is not a context-free language.

3. We say that a real number *x* is *Turing-constructible* if there is a Turing machine *M* that will write out the digits of *x* on its tape. For example, all rational numbers are Turing-constructible and so are irrational numbers *e*, *π*, √2. We can say that all numbers, for which there exists an algorithm to compute its value, are Turing-constructible. How many are the Turing-constructible real numbers? Justify your answer.

4. The Post Correspondence Problem (PCP) is said to have an *even solution* if there exists a sequence of **even** integers $i_1, i_2, ..., i_l$, where $t_{i1}t_{i2}... t_{il} = b_{i1}b_{i2}...b_{il}$. Thus, an even solution consists of only even-numbered domino tiles. Show that the problem of determining if an instance of the PCP has an even solution is undecidable.

5. Show that it is undecidable to determine if all the strings in $L(M)$, where M is a Turing machine, are of even length.

6. A *matching M* in an undirected graph $G = (V, E)$ is a set of edges, where no two edges share a common vertex. A *perfect matching* contains all the edges of G (thus the number of edges in a perfect matching is $|V|/2$). Let us define PERF-MATCH as follows:

PERF-MATCH = { <G> | the graph G has a perfect matching }

Show a polynomial time reduction of PERF-MATCH to SAT. In the reduction create an edge-variable $x_{(u,v)}$ for every edge (u,v) in G and build a Boolean formula that is true if and only if the edges (u,v) with $x_{(u,v)}=1$ form a complete matching.

a) Create the sub formula F_1 (not necessarily in conjunctive normal form) that guarantees that the set of edges is a matching (i.e. no two edges selected for each vertex).

b) Create the sub formula F_2 that guarantees that every vertex in G is in the matching.

c) Use the formulas F_1 and F_2 to construct a reduction from PERF-MATCH to SAT.