TÖL101F Algorithms, Logic and Complexity

Final exam **May 2nd, 2013**

Teacher: Hjálmtýr Hafsteinsson **Time: 9^{00} – 12^{00}**

All problems have the same value. ***You only have to solve 5 problems out of 6. The five best problems count.*** *All written material and a calculator are allowed.*

- Note that an answer without justification is worthless. Therefore justify all answers and remember that it is not necessary to write up definitions that are in the book.
- You can answer in either English or Icelandic.

1. Let A_1, A_2, ... be an enumerable collection of finite sets (i.e. each of the A_i's is finite). Is the union of all the sets enumerable or not? Justify your answer.

2. Let A and B be two infinite sets. A is enumerable, but B is not enumerable.

a) Show that $A \cup B$ is not enumerable.

b) Show that $A \cap B$ is enumerable.

c) What happens in *a)* and *b)* if A is also not enumerable? Justify your answers.

3. *a)* Can the following problem be proven undecidable **as a result of** the undecidability of the Halting problem: "For any Turing machine M there is a Turing machine H_M that can decide whethere M halts"? Justify your answer.

b) Show that it is undecidable if a Turing machine M halts for **at least one** input string.

4. *a)* Show an abacus machine to compute the function $f(x) = \lfloor \sqrt{x} \rfloor$, i.e. the largest integer $\leq$ the square root of x. You can use abacus machines from the textbook as blocks in your diagram. Also describe in words how your machine works.

b) Show that the following function is primitive recursive (by using previously defined primitive recursive functions):

$$f(x) = \begin{cases} \sqrt{x} & \text{if } \sqrt{x} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

5. A *counting quantifier* $\exists^{\geq n}$ can be added to first-order logic. The meaning of the formula $\exists^{\geq n} x F(x)$ is "there exist at least n elements x such that $F(x)$ holds".

a) Given the language of arithmetic L^* and its standard interpretation, show a formula using counting quantifiers that expresses the sentence "There are **at least 25** prime numbers less than 100".

b) Show that every formula using counting quantifiers can be converted into an equivalent formula that does not use counting quantifiers.

6. In chapter 11.1 we proved that the decision problem for logical implication was unsolvable. Based on that result show that the decision problem for satisfiability (i.e. "Given a finite set Γ of logical sentences, is Γ satisfiable?") is also unsolvable.