Displacement Current and Ampère-Maxwell Law

**The Ampère-Maxwell Law**

**Learning Goal:** To show that displacement current is necessary to make Ampère's law consistent for a charging capacitor.

Ampère's law relates the line integral of the magnetic field around a closed loop to the total current passing through that loop. This law was extended by Maxwell to include a new type of "current" that is due to changing electric fields: 

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{charge}} + I_{\text{displacement current}} \]

The first term on the right-hand side, \( I_{\text{charge}} \), describes the effects of the usual electric current due to moving charge. In this problem, that current is designated as usual. The second term, \( I_{\text{displacement current}} = \frac{d\mathbf{E}}{dt} \), is called the displacement current; it was recognized as necessary by Maxwell. His motivation was largely to make Ampère's law symmetric with Faraday's law of induction when the electric fields and magnetic fields are reversed. By calling for the production of a magnetic field due to a change in electric field, this law lays the groundwork for electromagnetic waves in which a changing magnetic field generates an electric field whose change, in turn, sustains the magnetic field. We will discuss these issues later. (Incidentally, a third type of "current," called magnetizing current, should also be added to account for the presence of changing magnetic materials, but it will be neglected, as it has been in the equation above.)

The purpose of this problem is to consider a classic illustration of the need for the additional displacement current term in Ampère's law. Consider the problem of finding the magnetic field that loops around just outside the circular plate of a charging capacitor. The cone-shaped surface shown in the figure has a current \( I(t) \) passing through it, so Ampère's law indicates a finite value for the field integral around this loop. However, a slightly different surface bordered by the same loop passes through the center of the capacitor, where there is no current due to moving charge. To get the same loop integral independent of the surface it must be true that either a current or an increasing electric field that passes through the Ampèrean surface will generate a looping magnetic field around its edge. The objective of this example is to introduce the displacement current, show how to calculate it, and then to show that the displacement current \( I_{\text{displacement current}} \) is identical to the conduction current \( I_{\text{charge}} \). Assume that the capacitor has plate area \( A \) and an electric field \( E(t) \) between the plates. Take \( \mu_0 \) to be the permeability of free space and \( \varepsilon_0 \) to be the permittivity of free space.

**Part A**

First find \( \oint \mathbf{B} \cdot d\mathbf{l} \), the line integral of \( \mathbf{B} \) around a loop of radius \( R \) located just outside the left capacitor plate. This can be found from the usual current due to moving charge in Ampère’s law, that is, without the displacement current.

Find an expression for this integral involving the current \( I(t) \) and any needed constants given in the introduction.

**ANSWER:**

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \text{Answer not displayed} \]

**Part B**

Now find an expression for \( \oint \mathbf{B} \cdot d\mathbf{l} \), the same line integral of \( \mathbf{B} \) around the same loop of radius \( R \) located just outside the left capacitor plate as before. Use the surface that passes between the plates of the capacitor, where there is no conduction current. This should be found by evaluating the amount of displacement current in the Ampère-Maxwell law above.

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \text{Answer not displayed} \]
Part B.1  Find the electric flux

Part B.2  Express \( \int \vec{B} \cdot d\vec{A} \) in terms of \( \Phi_E(t) \)

Express your answer in terms of the electric field between the plates \( E \), the plate area \( A \), and any needed constants given in the introduction.

\[
\text{ANSWER: } \int \vec{B} \cdot d\vec{A} = \text{Answer not displayed}
\]

A necessary consistency check

Part C
We now have two quite different expressions for the line integral of the magnetic field around the same loop. The point here is to see that they both are intimately related to the charge \( q(t) \) on the left capacitor plate. First find the displacement current \( J_{\text{displacement}}(t) \) in terms of \( q(t) \).

Part C.1  Find the flux using Gauss's law

Part C.2  Find the displacement current

Express your answer in terms of \( q(t) \), \( dq(t)/dt \), and any needed constants given in the introduction.

\[
\text{ANSWER: } J_{\text{displacement}}(t) = \text{Answer not displayed}
\]

Part D
Now express the normal current \( J_{\text{normal}}(t) \) in terms of the charge on the capacitor plate \( q(t) \).

Express your answer in terms of \( q(t) \), \( dq(t)/dt \), and any needed constants given in the introduction.

\[
\text{ANSWER: } J_{\text{normal}}(t) = \text{Answer not displayed}
\]

### The Magnetic Field in a Charging Capacitor

When a capacitor is charged, the electric field \( E \), and hence the electric flux \( \Phi_E \), between the plates changes. This change in flux induces a magnetic field, according to Ampère's law as extended by Maxwell:

\[
\int \vec{B} \cdot d\vec{A} = \mu_0 \left( I + \frac{\partial \Phi_E}{\partial t} \right)
\]

You will calculate this magnetic field in the space between capacitor plates, where the electric flux changes but the conduction current \( I \) is zero.

Part A
A parallel-plate capacitor of capacitance \( C \) with circular plates is charged by a constant current \( I \). The radius \( r \) of the plates is much larger than the distance \( d \) between them, so fringing effects are negligible. Calculate \( \vec{B}(\rho) \), the magnitude of the magnetic field inside the capacitor as a function of distance from the axis joining the center points of the circular plates.

\[
B(\rho) = \mu_0 C \left( \frac{1}{2\pi} \right) \rho^3 \frac{d}{d}\rho
\]

Express your answer in terms of \( \rho \) and given quantities.

\[
\text{ANSWER: } B(\rho) = \mu_0 C \left( \frac{1}{2\pi} \right) \rho^3 \frac{d}{d}\rho
\]
For each of the actions depicted, determine the direction (right, left, or zero) of the current induced to flow through the resistor in the circuit containing the secondary coil.

Part A
For the action depicted in the figure, in which direction does the current flow through the resistor?

Hint A.1 Magnetic flux
Magnetic flux is the product of the component of magnetic field perpendicular to a given area and the area itself. Conceptually, it is proportional to the number of magnetic field lines passing through a given area.

Hint A.2 Electromagnetic induction
Whenever magnetic flux through an area changes, an electromotive force (emf) is created around the area. This induced emf has a direction such that if a conductor is present, current will flow to create a secondary magnetic field that opposes the change in the original magnetic flux. Basically, the induced current will "try" to maintain the initial value of the magnetic flux.

Hint A.3 Magnetic field produced by a coil
The magnetic field produced by a coil has the same directional properties as the field produced by a single loop. To determine the direction of the magnetic field produced by a current in a coil, imagine grasping any loop of the coil with your right hand, with your thumb pointing in the direction of the current flow. The direction in which your fingers point, both inside and outside the coil, is the direction of the magnetic field in those regions.

Hint A.4 Change in magnetic field through the secondary coil
Initially, there is no magnetic field through the secondary coil. After the switch is closed, does the magnetic field that passes through the secondary coil point to the right or to the left?

ANSWER: left

Hint A.5 Induced magnetic field
Since the magnetic field in the secondary coil changed from zero to a leftward-directed field when the switch was closed, current will flow in the secondary loop to create a magnetic field oriented to the right to oppose this increase in flux. Now determine the direction in which current must flow through the resistor to create a field directed to the right.

ANSWER: right

Part B
For the action depicted in the figure, in which direction does the current flow through the resistor?

Part B.1 Change in magnetic field through the secondary coil
Before the switch is opened, does the magnetic field that passes through the secondary coil point to the right or to the left?

ANSWER: left

Part B.2 Induced magnetic field
Since the magnetic field in the secondary coil changed from a leftward-directed field to zero when the switch was opened, current will flow in the secondary loop to create a magnetic field oriented to the left to replace this decrease in flux. Now determine the direction in which current must flow through the resistor to create a field directed to the left.

ANSWER: left

Part C
For the action depicted in the figure, in which direction does the current flow through the resistor?
In Part D, the left coil (only) is moving to the left with velocity \( v \).

**Part D**

For the action depicted in the figure, in which direction does the current flow through the resistor?

**ANSWER:**
- right
- left
- zero

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**A Superconducting Cylinder**

At temperatures near absolute zero, the critical field \( \mu_0 H_c \) for vanadium, a type-I superconductor, approaches 0.142 tesla. The normal phase of vanadium has a magnetic susceptibility close to zero.

Consider a long, thin, solid vanadium cylinder with its axis parallel to an external magnetic field \( B_{ext} \) in the \( +z \) direction. At points far from the ends of the cylinder, by symmetry, all the magnetic vectors are parallel to the \( x \) axis.

First, consider an applied magnetic field \( B_{ext} = (0.130 \text{ T}) \hat{z} \).

**Part A**

At temperatures near absolute zero, what is the magnitude of the resultant magnetic field \( \vec{B} \) inside the cylinder (far from the ends)?

**Part A.1**

What is the critical field?

Choose the correct definition of the critical field \( \mu_0 H_c \).

**ANSWER:**
- the maximum magnitude of magnetic field into which an object can be placed and still exhibit superconductivity (at a given temperature below the critical temperature)
- the magnetic field due to coulomb (or conservative) forces.

When an object is in the superconducting phase, its internal magnetic field is zero.

Express your answer in teslas.

**ANSWER:**
\[ B_c = 0 \text{ T} \]

**Part B**

What is the magnitude of the resultant magnetic field \( \vec{B} \) outside the cylinder (far from the ends) for the situation described in the problem introduction?

**Part B.1**

Find the magnetic field due to the cylinder

Think of the magnetic field outside the cylinder as an applied field \( B_{ext} \) plus a magnetic field \( B_{ind} \) due to the cylinder. What is the magnitude of \( B_{ind} \)?

**Hint B.1a**

A solenoid analogy

If you give some thought to this problem, you may be able to solve it without performing any calculations. We know that the magnetic field inside the cylinder is zero. We know that the object is behaving as though there were a current flowing on its surface creating a magnetic field that exactly cancels the applied magnetic field. Because the applied magnetic field is parallel to the \( x \) axis, the induced magnetic field must also be parallel to the \( x \) axis.

Let's recap: We have a cylindrical object behaving as though a current were flowing along its surface in such a way that there is a magnetic field inside it parallel to its axis. Does this sound familiar? The cylinder is acting like a solenoid. Current flowing around the circumference produces a magnetic field inside that is parallel to the \( x \) axis (which is also the axis of the cylinder). Do you recall what the magnetic field is outside of a solenoid?

**ANSWER:**
\[ B_{ind} = 0 \]
Part C
What is the magnitude of the magnetization \( \vec{J}_i \) inside the cylinder? What is the magnitude of the magnetization \( \vec{J}_o \) outside the cylinder?

**Hint C.1** How to approach the problem
Magnetization \( \vec{J} \) is defined by the equation \( \vec{J} = \vec{H} - \mu_0 \vec{B} \), where \( \vec{H} \) is the total magnetic field and \( \mu_0 \vec{B} \) is the applied magnetic field. All of the vectors point along the \( x \) axis, so solving this equation is straightforward.

**Hint C.2** Value of \( \mu_0 \)
Recall that \( \mu_0 = 4\pi \times 10^{-7} \text{T m/A} \) in standard SI units.

Express your answers in amperes per meter separated by a comma.

**ANSWER:** \[ J_i = 0.130 \text{ T}, J_o = 1.03 \times 10^5 \text{ A/m} \]

Recall that there needs to be a material present to induce magnetization. In other words, the magnetization of free space is always zero.

Now consider an applied magnetic field \( \vec{B} = (0.200 \text{T}) \hat{z} \).

Part D
At temperatures near absolute zero, what is the magnitude of the resultant magnetic field \( \vec{B} \) inside the cylinder (far from the ends)?

**Hint D.1** How to approach the problem

**Hint D.2** More on superconductivity

**ANSWER:** \[ |\vec{B}| = 0.260 \text{ T} \]

Part E
What is the magnitude of the resultant magnetic field \( \vec{B} \) outside the cylinder (far from the ends)?

**Hint E.1** How to approach the problem

**ANSWER:** \[ |\vec{B}| = 0.260 \text{ T} \]

Part F
What is the magnitude of the magnetization \( \vec{J}_i \) inside the cylinder? What is the magnitude of the magnetization \( \vec{J}_o \) outside the cylinder?

Express your answers in amperes per meter separated by a comma.

**ANSWER:** \[ J_i = 0.0 \text{ A/m}, J_o = 0.0 \text{ A/m} \]