30. Inductance

Assignment is due at 2:00am on Wednesday, March 14, 2007
Credit for problems submitted late will decrease to 0% after the deadline has passed.
The wrong answer penalty is 2% per part. Multiple choice questions are penalized as described in the online help.
The unopened hint bonus is 2% per part.
You are allowed 4 attempts per answer.

Mutual Inductance

Mutual Inductance of a Double Solenoid

Learning Goal: To learn about mutual inductance from an example of a long solenoid with two windings.

To illustrate the calculation of mutual inductance it is helpful to consider the specific example of two solenoids that are wound on a common cylinder. We will take the cylinder to have radius \( \rho \) and length \( L \). Assume that the solenoid is much longer than its radius, so that its field can be determined from Ampère's law throughout its entire length:

\[
\oint B \cdot dl = \mu_0 I_{net}
\]

We will consider the field that arises from solenoid 1, which has \( n_1 \) turns per unit length. The magnetic field due to solenoid 1 passes (entirely, in this case) through solenoid 2, which has \( n_2 \) turns per unit length. Any change in magnetic flux from the field generated by solenoid 1 induces an EMF in solenoid 2 through Faraday's law of induction,

\[
\oint E \cdot dl = -\frac{d\Phi_t}{dt}
\]

Part A
Consider first the generation of the magnetic field by the current \( I_1(\tau) \) in solenoid 1. Within the solenoid (sufficiently far from its ends), what is the magnitude \( B_1(\tau) \) of the magnetic field due to this current?

ANSWER: \( B_1(\tau) = \frac{\mu_0 I_1(\tau)}{2\pi \rho} \)

Part B
Part not displayed

Part C
Part not displayed

Part D
Part not displayed

Part E
Part not displayed

Part F
Part not displayed

Self Inductance

Self-Inductance of a Solenoid

Learning Goal: To learn about self-inductance from the example of a long solenoid.

To explain self-inductance, it is helpful to consider the specific example of a long solenoid, as shown in the figure. This solenoid has only one winding, and so the EMF induced by its changing current appears across the solenoid itself. This contrasts with mutual inductance, where this voltage appears across a second coil wound on the same cylinder as the first.

Assume that the solenoid has radius \( R \), length \( L \) along the \( z \) axis, and is wound with \( n \) turns per unit length so that the total number of turns is equal to \( nL \). Assume that the solenoid is much longer than its radius.

As the current through the solenoid changes, the resulting magnetic flux through the solenoid will also change, and an electromotive force will be generated across the solenoid according to Faraday’s law of induction:

\[
\oint E \cdot dl = -\frac{d\Phi_t}{dt}
\]

Faraday’s law implies the following relation between the self-induced EMF across the solenoid and the current passing through it:
\[ E(t) = -L \frac{dI(t)}{dt} \]

The "direction of the EMF" is determined with respect to the direction of positive current flow, and represents the direction of the induced electric field in the inductor. This is also the direction in which the "back-current" that the inductor tries to generate will flow.

Part A
Suppose that the current in the solenoid is \( I(t) \). Within the solenoid, but far from its ends, what is the magnetic field \( B(t) \) due to this current?

**Express your answer in terms of \( I(t) \), quantities given in the introduction, and relevant constants (such as \( \mu_0 \)).**

**ANSWER:**

\[ B(t) = \mu_0 nI(t) \]

Note that this field is independent of the radial position (the distance from the axis of symmetry) as long as it is measured at a point well inside the solenoid.

Part B
What is the magnetic flux \( \Phi(t) \) through a single turn of the solenoid?

**Express your answer in terms of the magnetic field \( B(t) \), quantities given in the introduction, and any needed constants.**

**ANSWER:**

\[ \Phi(t) = B(t) \times \pi r^2 \]

Part C
Suppose that the current varies with time, so that \( dI(t)/dt \neq 0 \). Find the electromotive force \( E \) induced across the entire solenoid due to the change in current through the entire solenoid.

### Part C.1
Find the flux in terms of the current

In Part B you found the flux \( \Phi_1(t) \) through a single turn of the solenoid. Now find the flux \( \Phi_{\text{total}}(t) \) through the entire solenoid.

**Express your answer in terms of \( I(t) \), other quantities given in the introduction, and various constants such as \( \mu_0 \).**

**ANSWER:**

\[ \Phi_{\text{total}}(t) = \mu_0 n^2 I(t) \pi r^2 Z \]

### Part C.2
Find the EMF for the entire solenoid

You now have an expression for the magnetic flux that passes through the solenoid. From this, you should be able to derive an expression for the EMF in the solenoid. Suppose the total magnetic flux through the solenoid is \( \Phi_{\text{total}}(t) \). What is the electromotive force \( E \) generated in the solenoid by the changing flux \( \Phi_{\text{total}}(t) \)?

**Express your answer in terms of \( d\Phi_{\text{total}}(t)/dt \) and its derivative, and other variables given in the introduction.**

**ANSWER:**

\[ E = -\frac{d\Phi_{\text{total}}(t)}{dt} = -\mu_0 n^2 \frac{dI(t)}{dt} \pi r^2 Z \]

**Hint C.3**: Putting it together

You have now worked out three things:

1. the magnetic field from \( I(t) \);
2. the flux from this field;
3. the EMF for the entire solenoid.

Put them together and you have the answer!

**Express your answer in terms of \( dI(t)/dt \), \( r \), \( Z \), and \( I \).**

**ANSWER:**

\[ E = -\mu_0 n^2 \frac{dI(t)}{dt} \pi r^2 Z \]

Part D
The self-inductance \( L \) is related to the self-induced EMF \( E(t) \) by the equation \( E(t) = -L \frac{dI(t)}{dt} \). Find \( L \) for a long solenoid. (Hint: The self-inductance \( L \) will always be a positive quantity.)

**Express the self-inductance in terms of the number of turns per length \( n \), the physical dimensions \( r \) and \( Z \), and relevant constants.**

**ANSWER:**

\[ L = \mu_0 n^2 \pi r^2 Z \]

This definition of the inductance is identical to another definition you may have encountered: \( \Phi_M = I, L \), where \( \Phi_M \) is the magnetic flux due to a current \( I \) in the inductor. To see the correspondence you should differentiate both sides of this equation with respect to time and use Faraday's law, i.e., \( \frac{d\Phi_M}{dt} = -E \).

Now consider an inductor as a circuit element. Since we are now treating the inductor as a circuit element, we must discuss the voltage across it, not the EMF inside it. The important point is that the inductor is assumed to have no resistance. This means that the net electric field inside it must be zero when it is connected in a circuit. Otherwise, the current in it will become infinite. This means that the induced electric field \( E_{\text{in}} \) deposits charges on and around the inductor in such a way as to produce a nearly equal and opposite electric field \( E_{\text{out}} \) such that \( E_{\text{in}} = -E_{\text{out}} \).
Kirchhoff's loop law defines voltages only in terms of fields produced by charges (like $\vec{E}$), not those produced by changing magnetic fields (like $\vec{B}$). So if we wish to continue to use Kirchhoff's loop law, we must continue to use this definition consistently. That is, we must define the voltage $V_{BA} = V_A - V_B = \int_A^B \vec{E} \cdot d\vec{A}$ alone (note that the integral is from A to B rather than from B to A, hence the positive sign). So finally, $V_{BA} = \int_A^B \vec{E} \cdot d\vec{A} = \int_A^B (\vec{\nabla} \Phi + \vec{J}) \cdot d\vec{A} = \int_A^B \vec{\nabla} \Phi \cdot d\vec{A} = -\int_A^B \nabla \cdot \vec{E} \cdot d\vec{A} + \int_A^B \vec{J} \cdot d\vec{A}$, where we have used $\oint_A \vec{E} \cdot d\vec{A} = 0$ and the definition of $\mathcal{E}$. 

Part E
Which of the following statements is true about the inductor in the figure in the problem introduction, where $I(t)$ is the current through the wire?

**Hint E.1**
A fundamental inductance formula

**ANSWER:**
- If $I(t)$ is positive, the voltage at end A will necessarily be greater than that at end B.
- If $I(t)$ is positive, the voltage at end A will necessarily be greater than that at end B.
- If $I(t)$ is positive, the voltage at end A will necessarily be less than that at end B.
- If $I(t)$ is positive, the voltage at end A will necessarily be less than that at end B.

Part F
Now consider the effect that applying an additional voltage to the inductor will have on the current already flowing through it (imagine that the voltage is applied to end A, while end B is grounded). Which one of the following statements is true?

**Hint F.1**
A fundamental inductance formula

**ANSWER:**
- If $V(t)$ is positive, then $I(t)$ will necessarily be positive and $I(t)/dt$ will be negative.
- If $V(t)$ is positive, then $I(t)$ will necessarily be negative and $I(t)/dt$ will be positive.
- If $V(t)$ is positive, then $I(t)$ could be positive or negative, while $I(t)/dt$ will necessarily be negative.
- If $V(t)$ is positive, then $I(t)$ could be positive or negative while $I(t)/dt$ will necessarily be positive.
- If $V(t)$ is positive, then $I(t)$ will necessarily be negative and $I(t)/dt$ will be positive.
- If $V(t)$ is positive, then $I(t)$ will necessarily be negative and $I(t)/dt$ will be positive.

Note that when you apply Kirchhoff's rules and traverse the inductor in the direction of current flow, you are interested in $V_{BA} = V_B - V_A = -L(dI(t)/dt)$, just as traversing a resistor gives a term $-RI$.

In sum: when an inductor is in a circuit and the current is changing, the changing magnetic field in the inductor produces an electric field. This field opposes the change in current, but at the same time deposits charge, producing yet another electric field. The net effect of these electric fields is that the current changes, but not abruptly. The "direction of the EMF" refers to the direction of the first, induced, electric field.

### Mutual Inductance of a Tesla Coil

A long solenoid with cross-sectional area $A$ and length $L$ is wound with $N_a$ turns of wire. A shorter coil with $N_b$ turns of wire surrounds it.

**Part A**
Find the value $M$ of the mutual inductance.

**Hint A.1**
**Hint not displayed**

**Part A.2**
Find the magnetic field of the solenoid.

**Part not displayed**

**Hint A.3**
**The flux in coil 2**

**Hint not displayed**

**Hint A.4**
**Finding the mutual inductance**

**Hint not displayed**

**Express your answer in terms of $N_a, N_b, A, L$, and $\mu_0$**

**ANSWER:**

$M = \text{Answer not displayed}$

### Energy in Inductors and B-Fields

**Energy Stored in an Inductor**

The electric-power industry is interested in finding a way to store electric energy during times of low demand for use during peak-demand times. One way of achieving this goal is to use large...
Part A
What inductance \( L \) would be needed to store energy \( E = 3.0 \times 10^3 \text{ kwh} \) (kilowatt-hours) in a coil carrying current \( I = 200 \text{ A} \)?

**Hint A.1**

The formula for the energy stored in a current-carrying inductor

Recall the formula for energy stored in an inductor: \( U = \frac{LI^2}{2} \)

Part A.2
Express the energy in joules

What is the energy \( E = 3.0 \times 10^3 \text{ kwh} \) expressed in joules?

Express your answer numerically, to three significant figures.

ANSWER: \( E = 1.08 \times 10^7 \text{ joules} \)

This is probably not the best way to store energy: unless the coil is a superconductor, the amount of heat dissipated in the coil would be enormous. At this point, there is no way to produce large superconducting coils. Think of this problem as a practice exercise rather than a realistic example.

### The L-R Circuit

**A Parallel and Series LR Circuit Conceptual Question**

In the circuit shown in the figure, the two resistors are identical and the inductor is ideal (i.e., it has no resistance).

**Part A**
Is the current through \( I_1 \) greater than, less than, or equal to the current through \( I_2 \) immediately after the switch is first closed?

**Hint A.1**

Ideal inductors in circuits

Inductors have time-dependent effects on the behavior of electric circuits. When a potential difference is first applied to an ideal inductor, the inductor generates a back emf equal in magnitude to the potential difference applied, but opposite in direction. After a long time, the emf generated by the inductor falls to zero and an ideal inductor acts like a simple resistance-free wire.

**Hint A.2**

Replacing the inductor with an "open"

When the switch is first closed, the inductor generates a back emf equal in magnitude to the potential difference applied. This means that no net potential difference exists across the inductor, so no current can flow through the inductor. Thus, the inductor acts like an "open" in the circuit. Imagine simply removing the inductor from the circuit, leaving the circuit open at the location occupied by the inductor. Analyze the remaining circuit using ideas developed earlier.

ANSWER: greater than, less than, equal to, cannot be determined

**Part B**
Is the current through \( I_1 \) greater than, less than, or equal to the current through \( I_2 \) a very long time after the switch is closed?

**Hint B.1**

Replacing the inductor with a "short"

Long after the switch is closed, the inductor generates no emf and acts like a zero-resistance wire. Thus, the inductor acts like a "short" in the circuit. Imagine simply replacing the inductor with a bare wire. Analyze the remaining circuit using ideas developed earlier.

ANSWER: greater than, less than, equal to, cannot be determined

**Part C**
Is the current through \( I_1 \) greater than, less than, or equal to the current through \( I_2 \) immediately after the switch is opened (after being closed for a very long time)?

**Hint C.1**

Current without a battery

By opening the switch, the battery is removed from the circuit. If any current still flows in the circuit, it must be due to electromagnetic induction.

**Hint C.2**

Inductor as a battery

Since current flows through the inductor before the switch is opened, current will be induced in the inductor immediately after the switch is opened in an "attempt" to oppose the reduction in the magnetic flux due to the removal of the current. This current will flow from the inductor through any components that form a closed circuit with the inductor.
Decay of Current in an L-R Circuit

Learning Goal: To understand the mathematics of current decay in an L-R circuit

A DC voltage source is connected to a resistor of resistance $R$ and an inductor with inductance $L$, forming the circuit shown in the figure. For a long time before $t = t_0$, the switch has been in the position shown, so that a current $I_{t_0}$ has been built up in the circuit by the voltage source. At $t = t_0$, the switch is thrown to remove the voltage source from the circuit. This problem concerns the behavior of the current $I(t)$ through the inductor and the voltage $V(t)$ across the inductor at time $t$ after $t = t_0$.

Part A
After $t = t_0$, what happens to the voltage $V(t)$ across the inductor and the current $I(t)$ through the inductor relative to their values prior to $t = t_0$?

Part A.1 What is the relation between current and voltage for the inductor?

Part not displayed

ANSWER: Answer not displayed

Part B
What is the differential equation satisfied by the current $I(t)$ after time $t = t_0$?

Part B.1 Kirchhoff's loop law

Part not displayed

Express $\frac{dI(t)}{dt}$ in terms of $I(t)$, $R$, and $L$.

ANSWER: $\frac{dI(t)}{dt} = \text{Answer not displayed}$

Part C
What is the expression for $I(t)$ obtained by solving the differential equation that $I(t)$ satisfies after $t = t_0$?

Part C.1 Separation of variables

Part not displayed

Hint C.2 Integrating

Hint not displayed

Express your answer in terms of the initial current $I_{t_0}$ as well as $L$, $R$, and $t$.

ANSWER: $I(t) = \text{Answer not displayed}$

Part D
What is the time constant $\tau$ of this circuit?

Hint D.1 Definition of time constant

Hint not displayed

Hint D.2 The time constant for this circuit

Hint not displayed

Express your answer in terms of $I_{t_0}$ and $R$?

ANSWER: $\tau = \text{Answer not displayed}$

Growth of Current in an L-R Circuit
Learning Goal: To review the procedure for setting up and solving equations for determining current growth in an L-R circuit connected to a battery

Consider an L-R circuit as shown in the figure. The battery provides a voltage $V_b$. The inductor has inductance $L$, and the resistor has resistance $R$. The switch is initially open as shown. At time $t = 0$, the switch is closed.

**Part A**

What is the differential equation governing the growth of current in the circuit as a function of time after $t = 0$?

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**Part A.1**

Find the voltage change across the resistor

---

**Part A.2**

Find the voltage drop across the inductor

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**Part A.3**

Use Kirchhoff's loop law to sum the voltages

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Express the right-hand side of the differential equation for $\frac{dI(t)}{dt}$ in terms of $I(t)$, $V_b$, $R$, and $L$.

**ANSWER:**

$$\frac{dI(t)}{dt} = \frac{V_b}{L} - R \frac{I(t)}{L}$$

---

**Part B**

We know that the current in the circuit is growing. It will approach a steady state after a long time (as $t$ tends to infinity), which implies that the differential term in our circuit equation will tend to zero, hence simplifying our equation. The current around the circuit will tend to an asymptotic value $I_{\text{asymp}}$. What is $I_{\text{asymp}}$?

Express your answer in terms of $V_b$ and $R$ and other constants and variables given in the introduction.

**ANSWER:**

$$I_{\text{asymp}} = \frac{V_b}{R}$$

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**Part C**

Solve the differential equation obtained in Part A for the current $I(t)$ as a function of time after the switch is closed at $t = 0$.

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**Part C.1**

Substituting into the differential equation

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**Hint C.2**

A helpful integral

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**Part C.3**

Solving the differential equation 1: a substitution

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**Hint C.4**

Solving the differential equation 2: new limits

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Express your answer in terms of $V_b$, $R$, and $L$. Use the notation $\exp(x)$ for $e^x$.

**ANSWER:**

$$I(t) = \left( \frac{V_b}{R} \right) \left( 1 - \exp \left( - \left( \frac{R}{L} \right) t \right) \right)$$

The current approaches its asymptotic/steady-state value exponentially, i.e., usually very fast. However, the time constant depends on the actual values of the inductance and resistance.

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**Determining Inductance from the Decay in an L-R Circuit**

In this problem you will calculate the inductance of an inductor from a current measurement taken at a particular time. Consider the L-R circuit shown in the figure. Initially, the switch connects a resistor of resistance $R$ and an inductor to a battery, and a current $I_0$ flows through the circuit. At time $t = 0$, the switch is thrown open, removing the battery from the circuit. Suppose you measure that the current decays to $I_1$ at time $t_1$. 

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http://session.masteringphysics.com/myct
Part A
Determine the time constant \( \tau \) of the circuit.

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<th>How to approach the problem</th>
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<tr>
<th>Part A.2</th>
<th>What is the definition of time constant?</th>
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<th>Part A.3</th>
<th>Match the condition at ( t_1 )</th>
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<th>Hint A.4</th>
<th>Solve for the time constant ( \tau )</th>
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Express your answer in terms of \( t_1 \), \( I_L \), and \( I_i \). Recall that natural logarithms are entered as \( \ln(x) \).

ANSWER: \[ \tau = \text{Answer not displayed} \]

Part B
What is the inductance \( L \) of the inductor?

<table>
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<th>What is the time constant of an L-R circuit?</th>
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Express your answer in terms of \( R \), \( I_o \), \( I_f \), and \( t_f \).

ANSWER: \[ L = \text{Answer not displayed} \]

Power in a Decaying R-L Circuit

Consider an R-L circuit with a DC voltage source, as shown in the figure.

This circuit has a current \( I_L \) when \( t < 0 \). At \( t = 0 \) the switch is thrown removing the DC voltage source from the circuit. The current decays to \( I(t) \) at time \( t \).

Part A
What is the power, \( P_R(t) \), flowing into the resistor, \( R_1 \) at time \( t \)?

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<th>Ohm's Law</th>
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Express your answer in terms of \( I(t) \) and \( R_1 \).

ANSWER: \[ P_R(t) = \text{Answer not displayed} \]

Part B
What is the power flowing into the inductor?
Part B.1  Energy in an Inductor

Part B.2  Equation for $I(t)$

Hint B.3  Power into the Inductor

Express your answer as a function of $i$ and $I(t)$.

ANSWER: $P_L(t) = \text{Answer not displayed}$

Part C

The R-L Circuit: Responding to Changes

Learning Goal: To understand the behavior of an inductor in a series R-L circuit.

In a circuit containing only resistors, the basic (though not necessarily explicit) assumption is that the current reaches its steady-state value instantly. This is not the case for a circuit containing inductors. Due to a fundamental property of an inductor to mitigate any "externally imposed" change in current, the current in such a circuit changes gradually when a switch is closed or opened.

Consider a series circuit containing a resistor of resistance $R$ and an inductor of inductance $L$ connected to a source of emf $\mathcal{E}$ and negligible internal resistance. The wires (including the ones that make up the inductor) are also assumed to have negligible resistance.

Let us start by analyzing the process that takes place after switch $S_1$ is closed (switch $S_2$ remains open). In our further analysis, lowercase letters will denote the instantaneous values of various quantities, whereas capital letters will denote the maximum values of the respective quantities.

Note that at any time during the process, Kirchhoff's loop rule holds and is, indeed, helpful:

$$\mathcal{E} - v_R - v_L = 0.$$ 

Part A

Immediately after the switch is closed, what is the current in the circuit?

Hint A.1  How to approach the problem

ANSWER: $\frac{\mathcal{E}}{R}$

Part B

Immediately after the switch is closed, what is the voltage across the resistor?

Hint B.1  Ohm's law

ANSWER: $\mathcal{E}$

Part C

Immediately after the switch is closed, what is the voltage across the inductor?

Hint C.1  A formula for voltage across an inductor

ANSWER: $\mathcal{E}$

Part D

Shortly after the switch is closed, what is the direction of the current in the circuit?

ANSWER: clockwise
Part E
Shortly after the switch is closed, what is the direction of the induced EMF in the inductor?

**ANSWER:**
- clockwise
- counterclockwise
- There is no induced EMF because the initial value of the current is zero.

The induced EMF does not oppose the current; it opposes the change in current. In this case, the current is directed counterclockwise and is increasing; in other words, the time derivative of the counterclockwise current is positive, which is why the induced EMF is directed clockwise.

Part F
Eventually, the process approaches a steady state. What is the current in the circuit in the steady state?

**ANSWER:**
- \( \frac{E}{R} \)
- \( \frac{E}{R} - L \)
- \( \frac{E}{R} - L \)
- \( \frac{E}{R} + L \)
- \( \frac{E}{R} \)

Part G
What is the voltage \( v_L \) across the inductor in the steady state?

**ANSWER:**
- zero
- \( \frac{E}{2} \)
- \( \frac{E}{2} \)
- \( E - RL \)
- \( E + RL \)

**Hint G.1 Induced EMF and current**

**Hint not displayed**

Part H
What is the voltage \( v_R \) across the resistor in the steady state?

**ANSWER:**
- zero
- \( \frac{E}{2} \)
- \( \frac{E}{2} \)
- \( E - RL \)
- \( E + RL \)

Part I
Now that we have a feel for the state of the circuit in its steady state, let us obtain the expression for the current in the circuit as a function of time. Note that we can use the loop rule (going counterclockwise):

\[ E - v_R - v_L = 0. \]

Note as well that \( v_R = iR \) and \( v_L = L \frac{di}{dt} \). Using these equations, we can get, after some rearranging of the variables and making the substitution \( x = \frac{E}{R} - i \).

\[ \frac{dx}{x} = \frac{R}{L} \frac{di}{dt}. \]

Integrating both sides of this equation yields

\[ x = x_0 e^{-\frac{R}{L} t}. \]

Use this last expression to obtain an expression for \( i(t) \). Remember that \( x = \frac{E}{R} - i \) and that \( i_0 = i(0) = 0 \).
Express your answer in terms of $E$, $R$, and $L$. You may or may not need all these variables. Use the notation $\exp(x)$ for $e^x$.

**ANSWER:**

\[ i(t) = \frac{E}{R} \left( 1 - \exp\left( \frac{-Rt}{L} \right) \right) \]

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**Part J**

Just as in the case of $R\,C$ circuits, the steady state here is never actually reached: The exponential functions approach their limits asymptotically as $t \to \infty$. However, it usually does not take very long for the value of $i$ to get very close to its presumed limiting value. The next several questions illustrate this point.

Note that the quantity $L/R$ has dimensions of time and is called the time constant (you may recall similar terminology applied to $R\,C$ circuits). The time constant is often denoted by $\tau$. Using $\tau$, one can write the expression

\[ i(t) = \frac{E}{R} \left( 1 - \exp\left( \frac{-t}{\tau} \right) \right) \]

as

\[ i(t) = \frac{E}{R} \left( 1 - \exp\left( \frac{-t}{\tau} \right) \right) \]

Find the ratio of the current $i(t)$ at time $t = \frac{1}{2}\tau$ to the maximum current $I_{\text{max}} = \frac{E}{R}$.

**Express your answer numerically, using three significant figures.**

**ANSWER:**

\[ \frac{i(\frac{1}{2}\tau)}{I_{\text{max}}} = 0.998 \]

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**Part K**

Find the time $t$ it takes the current to reach 99.999% of its maximum value.

**Express your answer numerically, in units of $\tau$. Use three significant figures.**

**ANSWER:**

\[ t = 11.5 \ \tau \]

---

**Part L**

Find the time $t$ it takes the current to reach 99.999% of its maximum value. Assume that $R = 1\,\Omega$ and $L = 5\,\mu\text{H}$.

**Express your answer in seconds, using three significant figures.**

**ANSWER:**

\[ t = 5.76 \times 10^{-2} \ \text{seconds} \]

It does not take long at all! However, the situation can be different if the inductance is large and the resistance is small. The next example illustrates this point.

---

**Part M**

Find the time $t$ it takes the current to reach 99.999% of its maximum value. Assume that $R = 0.01\,\Omega$ and $L = 5\,\mu\text{H}$.

**Express your answer in seconds, using three significant figures.**

**ANSWER:**

\[ t = 5760 \ \text{seconds} \]

This is more than an hour and a half! Nobody would wait that long. Let us see what change would occur in such a circuit over a shorter period of time.

---

**Part N**

What fraction of the maximum value will be reached by the current one minute after the switch is closed? Again, assume that $R = 0.01\,\Omega$ and $L = 5\,\mu\text{H}$.

**Use three significant figures in your answer.**

**ANSWER:**

\[ \frac{i(1 \ \text{minute})}{I_{\text{max}}} = 0.113 \]

This is only 11.3% of the maximum value of the current!

Now consider a different situation. After switch $S_1$ has been closed for a long time, it is opened; simultaneously, switch $S_2$ is closed, as shown in the figure. This effectively removes the battery from the circuit.

The questions below refer to the time immediately after switch $S_1$ is opened and switch $S_2$ is closed.
Part O
What is the direction of the current in the circuit?

ANSWER:  
- clockwise
- counterclockwise
- The current is zero because there is no EMF in the circuit.

Part P
What is happening to the magnitude of the current?

ANSWER:  
- The current is increasing.
- The current is decreasing.
- The current remains constant.

Part Q
What is the direction of the EMF in the inductor?

ANSWER:  
- clockwise
- counterclockwise
- The EMF is zero because the current is zero.
- The EMF is zero because the current is constant.

Part R
Which end of the inductor has higher voltage (i.e., to which end of the inductor should the positive terminal of a voltmeter be connected in order to yield a positive reading)?

Hint R.1 How to approach the problem

ANSWER:  
- left
- right
- The potentials of both ends are the same.
- The answer depends on the magnitude of the time constant.

Part S
For this circuit, Kirchhoff's loop rule gives

\[ iR + L \frac{di}{dt} = \ell \]

Note that \( \frac{di}{dt} < 0 \) since the current is decreasing. Use this equation to obtain an expression for \( i(t) \).

Hint S.1 Initial and final conditions

Hint S.2 Let us cheat a bit

Express your answer in terms of \( \ell, L, \) and \( \ell \). Use \( \exp(x) \) for \( e^x \).

ANSWER:  
\[ i(t) = \frac{\ell}{L} \exp\left(-\frac{\ell t}{L}\right) \]

The L-C Circuit

Oscillations in an LC circuit.

Learning Goal: To understand the processes in a series circuit containing only an inductor and a capacitor.

Consider the circuit shown in the figure.

This circuit contains a capacitor of capacitance \( C \) and an inductor of inductance \( L \). The resistance of all wires is considered negligible.
Initially, the switch is open, and the capacitor has a charge \( q_0 \). The switch is then closed, and the changes in the system are observed. It turns out that the equation describing the subsequent changes in charge, current, and voltage is very similar to that of simple harmonic motion, studied in mechanics. To obtain this equation, we will use the law of conservation of energy.

Initially, the entire energy of the system is stored in the capacitor. When the circuit is closed, the capacitor begins to discharge through the inductor. As the charge of the capacitor decreases, so does its energy. On the other hand, as the current through the inductor increases, so does the energy stored in the inductor. Assuming no heat loss and no emission of electromagnetic waves, energy is conserved, and at any point in time, the sum of the energy stored in the capacitor \( U_C \) and the energy stored in the inductor \( U_L \) is a constant \( U \):

\[
U = U_C + U_L = \frac{q^2}{2C} + \frac{Lf^2}{2}
\]

where \( q \) is the charge on the capacitor and \( f \) is the current through the inductor (\( q \) and \( f \) are functions of time, of course). For this problem, take clockwise current to be positive.

**Part A**
Using the expression for the total energy of this system, it is possible to show that after the switch is closed,

\[
\frac{d^2q}{dt^2} = -kq,
\]

where \( k \) is a constant. Find the value of the constant \( k \).

**Part A.1**
The derivative of the total energy

**Part not displayed**

**Hint A.2**
The current in a series circuit

**Hint not displayed**

**Hint A.3**
Charge and current

**Hint not displayed**

Express your answer in terms of \( f \) and \( C \).

**ANSWER:**

\[ k = \frac{1}{L - C} \]

This expression can also be obtained by substituting \( f = -\frac{dq}{dt} \) into the equation obtained from Kirchhoff's loop rule. It is always good to solve a problem in more than one way!

**Part B**
From mechanics, you may recall that when the acceleration of an object is proportional to its coordinate,

\[
\frac{d^2x}{dt^2} = \frac{k}{m} = -\omega^2x,
\]

such motion is called simple harmonic motion, and the coordinate depends on time as \( x(t) = A \cos(\omega t + \phi) \), where \( \phi \) the argument of the harmonic function at \( t = 0 \) is called the phase constant.

Find a similar expression for the charge \( q(t) \) on the capacitor in this circuit. Do not forget to determine the correct value of \( \phi \) based on the initial conditions described in the problem.

**Hint B.1**
The phase constant

**Hint not displayed**

**Part B.2**
Find the period of the oscillations

**Part not displayed**

Express your answer in terms of \( q_0, L \), and \( C \). Use the cosine function in your answer.

**ANSWER:**

\[ q(t) = q_0 \cos \left( \frac{t}{\sqrt{L/C}} \right) \]

**Part C**
What is the current \( I(t) \) in the circuit at time \( t \) after the switch is closed?

**Hint C.1**
Current as a derivative of the charge

**Hint not displayed**

Express your answer in terms of \( q_0, L, C \), and other variables given in the introduction.

**ANSWER:**

\[ I(t) = \frac{q_0}{\sqrt{LC}} \sin \left( \frac{t}{\sqrt{L/C}} \right) \]
Part D
Recall that the top plate of the capacitor is positively charged at \( t = 0 \). In what direction does the current in the circuit begin to flow immediately after the switch is closed?

ANSWER:
- There is no current because there cannot be any current through the capacitor.
- Clockwise
- Counterclockwise

Part E
Immediately after the switch is closed, what is the direction of the EMF in the inductor? (Recall that the direction of the EMF refers to the direction of the back-current or the induced electric field in the inductor.)

ANSWER:
- There is no EMF until the current reaches its maximum.
- Clockwise
- Counterclockwise

In the remaining parts, assume that the period of oscillations is 8.0 milliseconds. Also, keep in mind that the top plate of the capacitor is positively charged at \( t = 0 \).

Part F
At what time does the current reach its maximum value for the first time?

Hint F.1
How to approach the problem

Hint F.2
Graph of \( I(t) \)

ANSWER:
- 0.0 ms
- 2.0 ms
- 4.0 ms
- 6.0 ms
- 8.0 ms

Part G
At what moment does the EMF become zero for the first time?

Hint G.1
Relationship between the induced EMF and the current.

Hint G.2
Graph of \( I(t) \)

ANSWER:
- 0.0 ms
- 2.0 ms
- 4.0 ms
- 6.0 ms
- 8.0 ms

Part H
At what moment does the current reverse direction for the first time?

Hint H.1
How to approach the problem

Hint H.2
Graph of \( I(t) \)

ANSWER:
- 0.0 ms
- 2.0 ms
- 4.0 ms
- 6.0 ms
- 8.0 ms

Part I
At what moment does the EMF reverse direction for the first time?

Hint I.1
Relationship between the induced EMF and the current.

Hint I.2
Graph of \( I(t) \)

ANSWER:
- 0.0 ms
- 2.0 ms
- 4.0 ms
- 6.0 ms
- 8.0 ms

Part J
At what moment does the energy stored in the inductor reach its maximum for the first time?

Hint J.1
Formula for the energy in an inductor
Part K
At what time do the capacitor and the inductor possess the same amount of energy for the first time?

**Hint K.1** Use conservation of energy

**Hint K.2** Energy and charge

**Hint K.3** A useful trigonometric result

Express your answer in milliseconds.

**ANSWER:** \[ t = 1.0 \text{ ms} \]

Part L
What is the direction of the current in the circuit 22.0 milliseconds after the switch is closed?

**Hint L.1** Use the periodic nature of the process

**Hint L.2** Graph of \( I(t) \)

**ANSWER:** The current is zero.

Part M
What is the direction of the EMF in the circuit 42.0 milliseconds after the switch is closed?

**Hint M.1** Graph of \( I(t) \)

**ANSWER:** The EMF is zero

---

**Energy within an L-C Circuit**

Consider an L-C circuit with capacitance \( C \), inductance \( L \), and no voltage source, as shown in the figure. As a function of time, the charge on the capacitor is \( Q(t) \) and the current through the inductor is \( I(t) \). Assume that the circuit has no resistance and that at one time the capacitor was charged.

---

**Part A**

As a function of time, what is the energy stored in the inductor?

Express your answer in terms of \( L \) and \( I(t) \).
### Part B
As a function of time, what is the energy $U_C(t)$ stored within the capacitor?

**Express your answer in terms of $C$ and $Q(t)$:**

**Answer:** $U_C(t) = \text{Answer not displayed}$

### Part C
What is the total energy $U_{\text{total}}$ stored in the circuit?

- **Part C.1** Charge and current as functions of time
  - Part not displayed

- **Part C.2** Find the resonance frequency
  - Part not displayed

- **Hint C.3** Summing the contributions
  - Hint not displayed

**Express your answer in terms of the maximum current $I_m$ and inductance $L$:**

**Answer:** $U_{\text{total}} = \text{Answer not displayed}$

Summary: 6 of 12 problems complete (50.29% avg. score) 27.18 of 27 points