Eðlisfræði 2, vor 2007

31. Alternating Current Circuits

Assignment is due at 2:00am on Wednesday, March 21, 2007

Credit for problems submitted late will decrease to 0% after the deadline has passed.
The wrong answer penalty is 2% per part. Multiple choice questions are penalized as described in the online help.
The unopened hint bonus is 2% per part.
You are allowed 4 attempts per answer.

Reactance and Phase

Voltage and Current in AC Circuits

Learning Goal: To understand the relationship between AC voltage and current in resistors, inductors, and capacitors, especially the phase shift between the voltage and the current.

In this problem, we consider the behavior of resistors, inductors, and capacitors driven individually by a sinusoidally alternating voltage source, for which the voltage is given as a function of time by $V(t) = V_0 \cos(\omega t)$. The main challenge is to apply your knowledge of the basic properties of resistors, inductors, and capacitors to these "single-element" AC circuits to find the current $I(t)$ through each. The key is to understand the phase difference, also known as the phase angle, between the voltage and the current. It is important to take into account the sign of the current, which will be called positive when it flows clockwise from the b terminal (which has positive voltage relative to the a terminal) to the a terminal (see figure). The sign is critical in the analysis of circuits containing combinations of resistors, capacitors, and inductors.

Part A
First, let us consider a resistor with resistance $R$ connected to an AC source (diagram 1). If the AC source provides a voltage $V_R(t) = V_0 \cos(\omega t)$, what is the current $I_R(t)$ through the resistor as a function of time?

Hint A.1  Ohm’s law

Express your answer in terms of $V_0$, $R$, $\omega$, and $t$.

\[
I_R(t) = \frac{V_0 \cos(\omega t)}{R}
\]

Note that the voltage and the current are in phase; that is, in the expressions for $V_R(t)$ and $I_R(t)$, the arguments of the cosine functions are the same at any moment of time. This will not be the case for the capacitor and inductor.

Part B
Now consider an inductor with inductance $L$ in an AC circuit (diagram 2). Assuming that the current in the inductor varies as $I_L(t) = I_0 \cos(\omega t)$, find the voltage $V_L(t)$ that must be driving the inductor.

Part B.1  Kirchhoff’s loop rule

Part B.2  The derivative of $\cos(\omega t)$

Part B.3  The phase relationship between sine and cosine

Express your answer in terms of $I_0$, $L$, $\omega$, and $t$. Use the cosine function, not the sine function, in your answer.

\[
V_L(t) = \frac{I_0 \cos(\omega t + \pi/2)}{\omega L}
\]

Graphs of $V_L(t)$ and $I(t)$ are shown below. As you can see, for an inductor, the voltage leads (i.e., reaches its maximum before) the current by $\frac{\pi}{2}$; in other words, the current lags the voltage by $\frac{\pi}{2}$. This can be conceptually understood by thinking of inductance as giving the current inertia: The voltage "tries" to push current through the inductor, but some sort of inertia resists the change in current. This is another manifestation of Lenz’s law. The difference $\frac{\pi}{2}$ is called the phase angle.
Part C

Again consider an inductor with inductance \( L \) connected to an AC source. If the AC source provides a voltage \( V_0(t) = V_0 \cos(\omega t) \), what is the current \( I_0(t) \) through the inductor as a function of time?

### Hint C.1 Using Part B

You can obtain the answer almost immediately if you consider the results of Part B: The amplitude of the voltage \( V_0 \) is \( L \omega I \); the frequency is the same as in Part B, and the phase difference is \( \frac{\pi}{2} \).

Do you remember what leads and what lags?

Express your answer in terms of \( V_0 \), \( L \), \( \omega \), and \( \varphi \). Use the cosine function, not the sine function, in your answer.

\[
I_0(t) = \frac{V_0 \cos(\omega t - \varphi)}{\sqrt{R^2 + (\omega L)^2}}
\]

For the amplitudes (magnitudes) of voltage and current, one can write \( V_0 = V_0 I \) (for the resistor) and \( V_0 = I_0 \omega L \) (for the inductor). If one compares these expressions, it should not come as a surprise that the quantity \( \omega L \), measured in ohms, is called inductive reactance; it is denoted by \( X_L \) (sometimes \( \varphi \)). It is called reactance rather than resistance to emphasize that there is no dissipation of energy. Using this notation, we can write \( V_0 = I_0 \varphi \) (for a resistor) and \( V_0 = I_0 \omega X_L \) (for an inductor). Also, notice that the current is in phase with voltage when a resistor is connected to an AC source; in the case of an inductor, the current lags the voltage by \( \frac{\pi}{2} \). What will happen if we replace the inductor with a capacitor? We will soon see.

Part D

Consider the potentials of points a and b on the inductor in diagram 2. If the voltage at point b is greater than that at point a, which of the following statements is true?

### Hint D.1 How to approach the problem

Try drawing graphs of the current through the inductor and voltage across the inductor as functions of time.

\[
\text{ANSWER:} \quad \begin{cases} \text{The current } I(t) \text{ must be positive (clockwise).} \\ \text{The current } I(t) \text{ must be directed counterclockwise.} \\ \text{The derivative of the current } \frac{dI(t)}{dt} \text{ must be negative.} \\ \text{The derivative of the current } \frac{dI(t)}{dt} \text{ must be positive.} \end{cases}
\]

It may help to think of the current as having inertia and the voltage as exerting a force that overcomes this inertia. This viewpoint also explains the lag of the current relative to the voltage.

Part E

Assume that at time \( t_{\text{max}} \), the current in the inductor is at a maximum; at that time, the current flows from point b to point a. At time \( t_{\text{max}} \), which of the following statements is true?

### Hint E.1 How to approach the problem

Try drawing graphs of the current through the inductor and voltage across the inductor as functions of time.

\[
\text{ANSWER:} \quad \begin{cases} \text{The voltage across the inductor must be zero and increasing.} \\ \text{The voltage across the inductor must be zero and decreasing.} \\ \text{The voltage across the inductor must be positive and momentarily constant.} \end{cases}
\]

Part F

Now consider a capacitor with capacitance \( C \) connected to an AC source (diagram 3). If the AC source provides a voltage \( V_0(t) = V_0 \cos(\omega t) \), what is the current \( I_C(t) \) through the capacitor as a function of time?

### Hint F.1 The relationship between charge and voltage for a capacitor

\[
\text{Hint not displayed}
\]

### Hint F.2 The relationship between charge and current

\[
\text{Hint not displayed}
\]

### Hint F.3 Mathematical details

\[
\text{Hint not displayed}
\]

Express your answer in terms of \( V_0 \), \( C \), \( \omega \), and \( t \). Use the cosine function with a phase shift, not the sine function, in your answer.
ANSWER: \[ I_C(t) = \frac{V_0}{X_C} \cos\left(\omega t + \frac{\pi}{2}\right) \]  \( \omega C \)

For the amplitude values of voltage and current, one can write \( V_0 = \frac{I_0}{\omega C} \). If one compares this expression with a similar one for the resistor, it should come as no surprise that the quantity \( \frac{1}{\omega C} \), measured in ohms, is called **capacitive reactance**; it is denoted by \( X_C \) (sometimes \( Z_C \)). It is called reactance rather than resistance to emphasize that there is no dissipation of energy. Using this notation, we can write \( V_0 = I_0 X_C \) and voltage lags current by \( \pi/2 \) radians (or 90 degrees). The notation is analogous to \( V_0 = I_0 R \) for a resistor, where voltage and current are in phase, and \( V_0 = I_0 X_L \) for an inductor, where voltage leads current by \( \pi/2 \) radians (or 90 degrees). We see, then, that in a capacitor, the voltage lags the current by \( \pi/2 \), while in the case of an inductor, the current lags the voltage by the same quantity \( \pi/2 \). In a capacitor, where voltage lags the current, you may think of the current as driving the change in the voltage.

### Part G
Consider the capacitor in diagram 3. Which of the following statements is true at the moment the alternating voltage across the capacitor is zero?

**Hint G.1** How to approach the problem
*Hint not displayed*

**Hint G.2** Graphs of \( V_C(t) \) and \( I(t) \)
*Hint not displayed*

### ANSWER:
- The current must be directed clockwise.
- The current must be directed counterclockwise.
- The current must be at a maximum.
- The current must be zero.

### Part H
Consider the capacitor in diagram 3. Which of the following statements is true at the moment the charge of the capacitor is at a maximum?

**Hint H.1** How to approach this problem
Since the voltage is directly proportional to the charge, when the charge is maximum, so is the voltage. Try drawing graphs of the (displacement) current through the capacitor and voltage across the capacitor as functions of time. Find the current when the voltage drop is maximum.

**Hint H.2** Graphs of \( V_C(t) \) and \( I(t) \)

### ANSWER:
- The current must be directed clockwise.
- The current must be directed counterclockwise.
- The current may be directed either clockwise or counterclockwise.
- The current must be zero.

### Part I
Consider the capacitor in diagram 3. Which of the following statements is true if the voltage at point b is greater than that at point a?

**Hint I.1** How to approach the problem
*Hint not displayed*

**Hint I.2** Graphs of \( V_C(t) \) and \( I(t) \)
*Hint not displayed*

### ANSWER:
- The current must be directed clockwise.
- The current must be directed counterclockwise.
- The current may be directed either clockwise or counterclockwise.
- The current must be zero.

### Part J
Consider a circuit in which a capacitor and an inductor are connected in parallel to an AC source. Which of the following statements about the magnitude of the current through the voltage source is true?

**Hint J.1** Driven AC parallel circuits
The voltage across each element is the same at every moment in time. However, the magnitudes of the currents in an AC circuit cannot be added without consideration of the phase angle between the
Inductive Reactance

**Learning Goal:** To understand the concept of reactance (of an inductor) and its frequency dependence.

When an inductor is connected to a voltage source that varies sinusoidally, a sinusoidal current will flow through the inductor, its magnitude depending on the frequency. This is the essence of AC (alternating current) circuits used in radio, TV, and stereo. Circuit elements like inductors, capacitors, and resistors are linear devices, so the amplitude \( I_0 \) of the current will be proportional to the amplitude \( V_0 \) of the voltage. However, the current and voltage may not be in phase with each other. This new relationship between voltage and current is summarized by the **reactance**, the ratio of voltage and current amplitudes, \( V_0 \) and \( I_0 \): 

\[
X_L = \frac{V_0}{I_0},
\]

where the subscript \( L \) indicates that this formula applies to an inductor.

**Part A**

To find the reactance \( X_L \) of an inductor, imagine that a current \( I(t) = I_0 \sin(\omega t) \) is flowing through the inductor. What is the voltage \( V(t) \) across this inductor?

**Part A.1** Voltage and current for an inductor

Express your answer in terms of \( I_0 \) and \( \omega \).

**ANSWER:**

\[
V(t) = \frac{V_0}{\sqrt{2}}
\]

**Part B**

**Part not displayed**

**Part C**

**Part not displayed**

Phasors and Examples

**Phasors: Analyzing a Parallel AC Circuit**

**Learning Goal:** To understand the use of phasors in analyzing a parallel AC circuit.

*Phasor diagrams, or simply phasors, provide a convenient graphical way of representing the quantities that change with time along with \( \cos(\omega t + \phi) \). This makes them useful for analyzing AC circuits with their inherent phase shifts between voltage and current. If a quantity \( I(t) \) changes with time as \( I(t) = I_0 \cos(\omega t + \phi) \), a phasor is a vector whose length represents the amplitude \( I_0 \) (see the diagram). This vector is assumed to rotate counterclockwise with angular speed \( \omega \); that way, the horizontal component of the vector represents the actual value \( I(t) \) at any given moment.*

In this problem, you will use the phasor approach to analyze an AC circuit. In answering the questions of this problem, keep the following in mind:

- For a resistor, the current and the voltage are always in phase.
- For an inductor, the current lags the voltage by \( \pi/2 \) radians.
- For a capacitor, the current leads the voltage by \( \pi/2 \) radians.

**Part A**

Phasors are helpful in determining the values of current and voltage in complex AC circuits.

Consider this phasor diagram: The diagram describes a circuit that contains two elements connected in parallel to an AC source. The vector labeled \( I_0 \) corresponds to the voltage across both elements of the circuit. Based on the diagram, what elements can the circuit contain?
Phasors Explained

**Learning Goal:** To understand the concept of phasor diagrams and be able to use them to analyze AC circuits (those with sinusoidally varying current and voltage).

Phasor diagrams provide a convenient graphical way of representing the quantities that change with time along with $\cos(\omega t + \phi)$, which makes such diagrams useful for analyzing AC circuits with their inherent phase shifts between voltage and current. You have studied the behavior of an isolated resistor, inductor, and capacitor connected to an AC source. However, when a circuit contains more than one element (for instance, a resistor and a capacitor or a resistor and an inductor or all three elements), phasors become a useful tool that allows us to calculate currents and voltages rather easily and also to visualize some important processes taking place in the AC circuit, such as resonance.

Let us assume that a certain quantity $f(t)$ changes over time as $f(t) = I_0 \cos(\omega t)$. A phasor is a vector whose length represents the amplitude $I_0$ (see the diagram). This vector is assumed to rotate counterclockwise with angular frequency $\omega$; that way, the horizontal component of the vector represents the actual value of $f(t)$ at any given moment.

In this problem, you will answer some basic questions about phasors and prepare to use them in the analysis of various AC circuits.

In parts A - C consider the four phasors shown in the diagram. Assume that all four phasors have the same angular frequency $\omega$.

**Part A**

At the moment $t$ depicted in the diagram, which of the following statements is true?

- $I_2$ leads $I_1$ by $\pi$.
- $I_3$ leads $I_2$ by $\pi$.

**Part B**

**Part C**

Find the amplitude of the current $I_s$ through the voltage source.

**Hint C.1**
The parallel connection

**Hint C.2**
The current through the voltage source as a vector sum

Express your answer in terms of the magnitudes of the individual currents $I_1$ and $I_2$.

**ANSWER:** $= \text{Answer not displayed}$

**Part D**

What is the tangent of the phase angle $\phi$ between the voltage and the current through the voltage source?

**Hint D.1**
The current through the voltage source as a vector sum

Express $\tan(\phi)$ in terms of $I_1$ and $I_2$.

**ANSWER:** $= \text{Answer not displayed}$
Part B
At the moment shown in the diagram, which of the following statements is true?

ANSWER:
- $I_2$ leads $I_1$ by $\frac{\pi}{2}$.
- $I_1$ leads $I_2$ by $\frac{\pi}{2}$.

Part C
At the moment shown in the diagram, which of the following statements is true?

ANSWER:
- $I_2$ leads $I_1$ by $\frac{\pi}{2}$.
- $I_1$ leads $I_2$ by $\frac{\pi}{2}$.

Let us now consider some basic applications of phasors to AC circuits.

- For a resistor, the current and the voltage are always in phase.
- For an inductor, the current lags the voltage by $\frac{\pi}{2}$.
- For a capacitor, the current leads the voltage by $\frac{\pi}{2}$.

Part D
Consider this diagram. Let us assume that it describes a series circuit containing a resistor, a capacitor, and an inductor. The current in the circuit has amplitude $I$, as indicated in the figure.

Which of the following choices gives the correct respective labels of the voltages across the resistor, the capacitor, and the inductor?

ANSWER:
- $V_1$, $V_2$, $V_3$.
- $V_1$, $V_2$, $V_3$.
- $V_1$, $V_2$, $V_3$.
- $V_1$, $V_2$, $V_3$.

Part E
Now consider a diagram describing a parallel AC circuit containing a resistor, a capacitor, and an inductor. This time, the voltage across each of these elements of the circuit is the same; on the diagram, it is represented by the vector labeled $V_0$.

The currents in the resistor, the capacitor, and the inductor are represented respectively by which vectors?

ANSWER:
- $I_1$, $I_2$, $I_3$.
### A Resistor and a Capacitor in a Series AC Circuit

A resistor with resistance $R$ and a capacitor with capacitance $C$ are connected in series to an AC voltage source. The time-dependent voltage across the capacitor is given by $V_C(t) = V_0 \sin \omega t$.

**Part A**
What is the amplitude $I_0$ of the total current $I(t)$ in the circuit?

<table>
<thead>
<tr>
<th>Hint A.1</th>
<th>How to approach the problem</th>
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<td>Hint not displayed</td>
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<table>
<thead>
<tr>
<th>Hint A.2</th>
<th>Applying Ohm's law to a capacitor</th>
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<table>
<thead>
<tr>
<th>Hint A.3</th>
<th>The reactance of a capacitor</th>
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</table>

Express your answer in terms of any or all of $R$, $C$, $V_0$, and $\omega$.

**ANSWER:**

$\text{Answer not displayed}$

**Part B**
Part not displayed

**Part C**
Part not displayed

### Determining Inductance from Voltage and Current

An inductor is hooked up to an AC voltage source. The voltage source has EMF $V_0$ and frequency $f$. The current amplitude in the inductor is $I_0$.

**Part A**
What is the reactance $X_L$ of the inductor?

<table>
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<tr>
<th>Hint A.1</th>
<th>Definition of reactance</th>
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<tbody>
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</table>

Express your answer in terms of $V_0$ and $I_0$.

**ANSWER:**

$X_L = \frac{V_0}{I_0}$

**Part B**
What is the inductance $L$ of the inductor?

<table>
<thead>
<tr>
<th>Hint B.1</th>
<th>Reactance in terms of $I$ and $\omega$</th>
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Express your answer in terms of $V_0$, $f$, and $I_0$.

**ANSWER:**

$L = \frac{V_0}{I_0 f}$

**Part C**
What would happen to the amplitude of the current in the inductor if the inductance $L$ were doubled?

<table>
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<tr>
<th>Hint C.1</th>
<th>How to approach the problem</th>
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**ANSWER:**

○ There would be no change in the amplitude of the current.
○ The amplitude of the current would be doubled.
○ The amplitude of the current would be halved.
○ The amplitude of the current would be quadrupled.
A Driven Series L-C Circuit

Learning Goal: To understand why a series L-C circuit acts like a short circuit at resonance.

An AC source drives a sinusoidal current of amplitude \( I_0 \) and frequency \( \omega \) into an inductor having inductance \( L \) and a capacitor having capacitance \( C \) that are connected in series. The current as a function of time is given by \( I(t) = I_0 \sin(\omega t) \).

Part A
Recall that the voltages \( V_L(t) \) and \( V_C(t) \) across the inductor and capacitor are not in phase with the respective currents \( I_L(t) \) and \( I_C(t) \). In particular, which of the following statements is true for a sinusoidal current driver?

ANSWER:
- \( V_L(t) \) and \( V_C(t) \) both lag their respective currents.
- \( V_L(t) \) and \( V_C(t) \) both lead their respective currents.
- \( V_L(t) \) lags \( I_L(t) \) and \( V_C(t) \) leads \( I_C(t) \).
- \( V_L(t) \) leads \( I_L(t) \) and \( V_C(t) \) lags \( I_C(t) \).

The phase angle between voltage and current for inductors and capacitors is 90 degrees, or \( \frac{\pi}{2} \) radians. Among other things, this means that no power is dissipated in either the inductor or the capacitor, since the time average of current times voltage, \( \langle I(t)V(t) \rangle \), is zero.

Part B
What is \( V(t) \), the voltage delivered by the current source?

Hint B.1 Current in a series circuit
Note that the current through the current source, the capacitor, and the inductor are all equal at all times.

Part B.2 Find the voltage across the capacitor
What is \( V_C(t) \), the voltage across the capacitor as a function of time?

Part B.2.a Find the charge on the capacitor
What is the charge \( q(t) \) on the capacitor?

ANSWER:
\[ q(t) = CV_C(t) \]

Part B.2.b What is the current in terms of the charge on the capacitor?
With the conventions in the circuit diagram, what is the current \( I_C(t) \) on the capacitor in terms of its voltage?

Express your answer in terms of the capacitance \( C \) and the voltage \( V_C(t) \) across it.

ANSWER:
\[ I_C(t) = \frac{dV_C(t)}{dt} \]

Integrate both sides of the equation \( I_C(t) = I_0(t) = C \frac{dV_C(t)}{dt} \) (once you substitute in the appropriate expression for \( I_C(t) \)). The constant of integration is zero because there is no average (DC) current or voltage in a pure AC circuit.

Express your answer in terms of \( I_L \), \( \omega \), and \( t \).

\[ V_C(t) = \frac{I_0}{C \omega} \cos(\omega t) \]

Part B.3 What is the voltage across the inductor?
What is \( V_L(t) \), the voltage across the inductor as a function of time?

Hint B.3.a Voltage and current for an inductor

\[ V_L(t) = \frac{I_0}{\omega} \sin(\omega t) \]

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Express \( V_L(t) \) in terms of \( I_L \omega \) and \( f \).

**ANSWER:**

\[
V_L(t) = I_L \omega L \cos(\omega t)
\]

**Hint R.4 Total voltage**

You should have defined the voltages across the capacitor and inductor in such a way that the total voltage \( V_{total}(t) \) is equal to \( V_C(t) + V_L(t) \), that is, so that total voltage is obtained by adding the voltages at time \( t \), with careful attention paid to the signs.

Express \( V(t) \) in terms of some or all of the variables \( I_L, X_L = \omega L \) and \( X_C = \frac{1}{\omega C} \), or \( C \) and \( L, \omega \), and, of course, time \( t \).

**ANSWER:**

\[
V(t) = I_L \omega L \cos(\omega t) - I_L \frac{1}{(C \omega)} \cos(\omega t)
\]

**Part C**

With \( V_L \) and \( V_C \), the amplitudes of the voltages across the inductor and capacitor, which of the following statements is true?

- At very high frequencies \( V_C \gg V_L \) and at very low frequencies \( V_C \ll V_L \)
- At very high frequencies \( V_L \gg V_C \) and at very low frequencies \( V_L \ll V_C \)
- \( V_C \gg V_L \) for all frequencies.
- \( V_L \ll V_C \) for all frequencies.
- \( V_C \) and \( V_L \) are about the same at all frequencies.

**Part D**

The behavior of the \( L-C \) circuit provides one example of the phenomenon of resonance. The resonant frequency is \( \omega_0 = \frac{1}{\sqrt{LC}} \). At this frequency, what is the amplitude \( V_C \) of the voltage supplied by the current source?

**ANSWER:**

\( V_C = 0 \)

**Part E**

Which of the following statements best explains this fact that at the resonant frequency, there is zero voltage across the capacitor and inductor?

- The voltage \( V(t) \) is zero at all times because \( V_L(t) = -V_C(t) \).
- The voltages \( V_L(t) \); \( V_C(t) \); and \( V_{total}(t) \) are zero at all times.
- The voltage \( V(t) \) is zero only when the current is zero.
- The voltage \( V(t) \) is zero only at times when the current is stationary (at a max or min).

At the resonant frequency of the circuit, the current source can easily push the current through the series \( L-C \) circuit, because the circuit has no voltage drop across it at all! Of course, there is always a voltage across the inductor, and there is always a voltage across the capacitor, since they do have a current passing through them at all times; however, at resonance, these voltages are exactly out of phase, so that the net effect is a current passing through the capacitor and the inductor without any voltage drop at all. The \( L-C \) series circuit acts as a short circuit for AC currents exactly at the resonant frequency. For this reason, a series \( L-C \) circuit is used as a trap to conduct signals at the resonant frequency to ground.

**A Voltage-Driven Parallel L-C Circuit**

An AC source that provides a voltage \( V(t) = V_{source} \sin(\omega t) \) drives an inductor having inductance \( L \) and a capacitor having capacitance \( C \), all connected in parallel.

**Part A**

Recall that the currents \( I_L(t) \) and \( I_C(t) \) through the inductor and capacitor are not in phase with their respective voltages \( V_L(t) \) and \( V_C(t) \). In particular, for a sinusoidal voltage driver, which of the following statements is true?

**ANSWER:**

Answer not displayed

**Part B**

Recall that the currents \( I_L(t) \) and \( I_C(t) \) through the inductor and capacitor are not in phase with their respective voltages \( V_L(t) \) and \( V_C(t) \). In particular, for a sinusoidal voltage driver, which of the following statements is true?
What is the amplitude of the current through the voltage source? In other words, if the time-dependent current is \( I(t) = I_0 \sin(\omega t) \), what is the value of \( I_0 \)?

**Hint B.1** Voltage in a parallel circuit

**Hint not displayed**

**Part B.2** Current through the capacitor

**Part not displayed**

**Part B.3** The current through the inductor

**Part not displayed**

**Hint B.4** Total current

**Hint not displayed**

Express the amplitude of the current \( I_0 \) in terms of \( V_0 \), \( C \), \( L \), and \( \omega \).

**ANSWER:**

\[ I_0 = \text{Answer not displayed} \]

**Part C**

With \( I_L(t) \) and \( I_C(t) \) representing the respective magnitudes of the currents through the inductor and capacitor, which of the following statements is true?

**ANSWER:**

\[ I(t) = \text{Answer not displayed} \]

**Part D**

The L-C circuit is one example of a system that can exhibit resonance behavior. The resonant frequency is \( \omega_0 = \frac{1}{\sqrt{LC}} \). At this frequency, what is the amplitude of the current supplied by the voltage source?

**Express your answer in terms of \( V_0 \), and any other need terms from the problem introduction.**

**ANSWER:**

\[ I(t) = \text{Answer not displayed} \]

**Part E**

**Reactance and Current**

Consider the two circuits shown in the figure. The current in circuit 1, containing an inductor of self-inductance \( L \), has an angular frequency \( \omega_1 \) while the current in circuit 2, containing a capacitor of capacitance \( C \), has an angular frequency \( \omega_2 \). If we increase \( \omega_1 \) and decrease \( \omega_2 \), both bulbs grow dimmer.

**Part A**

If we keep \( \omega_1 \) and \( \omega_2 \) constant, we can achieve the exact same effect of decreasing the brightness of each bulb by performing which of the following sets of actions?

**Hint A.1** How to approach the problem

**Hint not displayed**

**Hint A.2** Inductive reactance

**Hint not displayed**

**Hint A.3** Capacitive reactance

**Hint not displayed**

**Part A.4** Determine how \( X_L \) can be changed

**Part not displayed**

**Part A.5** Determine how \( X_C \) can be changed

**Part not displayed**

**ANSWER:**

1. \( X_L \) increasing
2. \( X_C \) decreasing
3. Both \( X_C \) and \( X_L \) increasing
4. Both \( X_C \) and \( X_L \) decreasing

As you found out, the reactance of these circuits can be changed not only by varying the frequency of the current, but also by changing the characteristics of the elements in them, i.e., by changing...
Part B
Now combine the capacitor, the inductor, and the bulbs in a single circuit, as shown in the figure. What happens to the brightness of each bulb if you increase the frequency of the current in the new circuit while keeping \( L \) and \( C \) constant?

![Diagram of a circuit with a capacitor, an inductor, and two bulbs]

**Part B.1  How to approach the problem**
In the new circuit, bulb 1 is still in series with the inductor, and so is bulb 2 with respect to the capacitor. Therefore, to determine how their brightness may change, you need to analyze how the current in both the inductor and the capacitor may change at higher frequencies. Note that because the branch of the circuit containing the inductor and bulb 1 is in parallel with the branch containing the capacitor and bulb 2, if the current in one branch decreases, the current in the other branch increases.

**Part B.2  Find which element experiences a decrease in current at higher frequencies**
Which element of the circuit will experience a decrease in current at higher frequencies?

**Hint B.2.a  Current in circuit elements**

**Hint B.2.b  Inductive reactance**

**Hint B.2.c  Capacitive reactance**

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**ANSWER:** the inductor

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Since inductive reactance is proportional to frequency, for a given voltage, high-frequency currents will have a much smaller amplitude through the inductor than through the capacitor. That is, the inductor tends to block high-frequency currents. The opposite situation occurs if the frequency is decreased. The capacitor will block low-frequency currents and bulb 2 will grow dimmer.

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**A High-Pass Filter**

A series \( L-R-C \) circuit consisting of a voltage source, a capacitor of capacitance \( C \), an inductor of inductance \( L \), and a resistor of resistance \( R \) is driven with an AC voltage of amplitude \( V_{in} \) and frequency \( \omega \). Define \( V_{out} \) to be the amplitude of the voltage across the resistor and the inductor.

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**Part A**
Find the ratio \( \frac{V_{out}}{V_{in}} \)

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**Part A.1  Find \( V_{in} \)**
Assume that the amplitude of the current in the circuit is \( I \). Write down an equation for \( V_{in} \), the amplitude of the input voltage of the circuit.

**Hint A.1.a  How to approach the problem**

**Hint A.1.b  Combined impedance**

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Express your answer in terms of \( I \), \( R \), the reactance \( X_C \) of the capacitor, and the reactance \( X_L \) of the inductor.
Part A.2  Find $V_{\text{out}}$

Assuming that the amplitude of the current in the circuit is $I$, write down an equation for the output voltage $V_{\text{out}}$ of the circuit.

Express your answer in terms of $I$, $R$, the reactance of the capacitor $X_C$, and the reactance of the inductor $X_L$.

ANSWER: \[ V_{\text{out}} = I \left( X_L^2 + R^2 \right) \]

Express your answer in terms of either $R$, $\omega$, $L$ and $C$, or $R$, $X_L = \omega L$, and $X_C = \frac{1}{\omega C}$.

ANSWER: \[ \frac{V_{\text{out}}}{V_{\text{in}}} = \left( \frac{1}{\sqrt{X_L^2 + R^2}} \right) \left( \frac{1}{\sqrt{X_C^2 + R^2}} \right) \]

For the following questions it will be useful to write the voltage ratio in the following form:

\[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1 + \left( \omega L/R \right)^2}{1 + \left( \omega L/R - 1/(\omega RC) \right)^2} \]

Part B

Which of the following statements is true in the limit of large $\omega$ ($\omega \gg \frac{1}{RC}$, $\frac{1}{\sqrt{LC}}$, $\frac{R}{L}$)?

Hint B.1  Implications of large $\omega$

In the large $\omega$ limit, $\omega L/R \gg 1/(\omega RC)$, so you can approximate $\omega L/R - 1/(\omega RC) \approx \omega L/R$.

ANSWER: $\frac{V_{\text{out}}}{V_{\text{in}}}$ is proportional to $\frac{1}{\omega}$. 

$\frac{V_{\text{out}}}{V_{\text{in}}}$ is proportional to $\frac{1}{\omega^2}$. 

$\frac{V_{\text{out}}}{V_{\text{in}}}$ is proportional to $\omega$. 

$\frac{V_{\text{out}}}{V_{\text{in}}}$ is close to 1.

Part C

Which of the following statements is true in the limit of small $\omega$ ($\omega \ll \frac{1}{RC}$, $\frac{1}{\sqrt{LC}}$, $\frac{R}{L}$)?

Hint C.1  Implications of small $\omega$

In the small $\omega$ limit, $\omega L/R \ll 1/(\omega RC)$, so you can approximate $\omega L/R - 1/(\omega RC) \approx -1/(\omega RC)$, and $\omega L/R - 1$ and $1/(\omega RC) \gg 1$.

ANSWER: $\frac{V_{\text{out}}}{V_{\text{in}}}$ is proportional to $\frac{1}{\omega}$. 

$\frac{V_{\text{out}}}{V_{\text{in}}}$ is proportional to $\frac{1}{\omega^2}$. 

$\frac{V_{\text{out}}}{V_{\text{in}}}$ is close to 1.

When $\omega$ is large, $\frac{V_{\text{out}}}{V_{\text{in}}} \approx \frac{1}{\omega}$, and when $\omega$ is small, $\frac{V_{\text{out}}}{V_{\text{in}}} \propto \omega$. Therefore, this circuit has the property that only the amplitude of the low-frequency inputs will be attenuated (reduced in value) at the output, while the amplitude of the high-frequency inputs will pass through relatively unchanged. This is why such a circuit is called a high-pass filter.

Constructing a Low-Pass Filter

A series $L-R-C$ circuit is driven with AC voltage of amplitude $V_{\text{in}}$ and frequency $\omega$. Define $V_{\text{out}}$ to be the amplitude of the voltage across the capacitor. The resistance of the resistor is $R$, the capacitance of the capacitor is $C$, and the inductance of the inductor is $L$.
Part A

What is the ratio \( \frac{V_{\text{out}}}{V_{\text{in}}} \)?

**Part A.1** Find \( V_{\text{in}} \)

**Part A.2** Find \( V_{\text{out}} \)

Express your answer in terms of either \( j \), \( \omega \), \( L \), and \( C \) or \( j \), \( X_C \) and \( X_L \).

**ANSWER:** \( \frac{V_{\text{out}}}{V_{\text{in}}} = \text{Answer not displayed} \)

Part B

Part not displayed

Part C

Part not displayed

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**Power in ac Circuits**

**Alternating Current, LC circuit**

A capacitor with capacitance \( C \) is connected in parallel to two inductors: inductor 1 with inductance \( L_1 \) and inductor 2 with inductance \( 2L_2 \), as shown in the figure. The capacitor is charged up to a voltage \( V \); at which point it has a charge \( \varphi \). There is no current in the inductors. Then the switch is closed.

---

**Part A**

Since the two inductors are in parallel, the voltage across them is the same at any time. Hence, \( V_1 = V_2 = X_1I_1 = X_2I_2 \) where \( X_1 \) and \( X_2 \) are the reactances of inductors 1 and 2, and \( I_1 \) and \( I_2 \) are the currents through them. Use this equality to express \( I_1 \) in terms of \( I_2 \).

**Hint A.1** The reactance of an inductor

**Express your answer in terms of \( I_2 \).**

**ANSWER:** \( I_1 = \text{Answer not displayed} \)

**Part B**

What is the effective inductance \( L_{\text{eff}} \) of the inductors 1 and 2 in the circuit?

**Hint B.1** Formulas for effective inductance

**Express your answer in terms of \( I_1 \).**

**ANSWER:** \( L_{\text{eff}} = \text{Answer not displayed} \)

**Part C**

Find the maximum current \( I_{\text{max}} \) through inductor 1.
Part C.1
Use conservation of energy to find the total maximum current.

Express your answer in terms of $C$, $I_x$, and $V$.

ANSWER: $I_{\text{max}} = \text{Answer not displayed}$

Alternating Voltages and Currents

The voltage supplied by a wall socket varies with time, reversing its polarity with a constant frequency, as shown in the graph.

Part A
What is the rms value $V_{\text{rms}}$ of the voltage plotted in the graph?

**Hint A.1**
RMS value of a quantity with sinusoidal time dependence

A quantity that varies with time as $x = x_{\text{max}} \sin(\omega t)$ (or $x = x_{\text{max}} \cos(\omega t)$) has a maximum value equal to $x_{\text{max}}$ and an rms value given by $x_{\text{rms}} = x_{\text{max}} / \sqrt{2}$.

Part A.2
Find the maximum value of the voltage $V_{\text{max}}$.

What is the maximum value $V_{\text{max}}$ of the voltage plotted in the graph?

Express your answer in volts.

ANSWER: $V_{\text{max}} = 220 \text{ V}$

Express your answer in volts.

ANSWER: $V_{\text{rms}} = 120 \text{ V}$

This is the standard rms voltage supplied to a typical household in North America.

Part B
When a lamp is connected to a wall plug, the resulting circuit can be represented by a simplified AC circuit, as shown in the figure. Here the lamp has been replaced by a resistor with an equivalent resistance $R = 120 \Omega$. What is the rms value $I_{\text{rms}}$ of the current flowing through the circuit?

**Hint B.1**
Ohm's law in AC circuits

Ohm's law can still be applied to an AC circuit, provided the values used to describe all physical quantities are consistent. For example, Ohm's law can be written using maximum values of voltage and current, or alternatively using rms quantities.

Express your answer in amperes.

ANSWER: $I_{\text{rms}} = 1.00 \text{ A}$

Part C
What is the average power $P_{\text{avg}}$ dissipated in the resistor?

**Hint C.1**
Average power in an AC circuit

Hint not displayed
Express your answer in watts.

\[ P_{\text{avg}} = 120 \text{ W} \]

The instantaneous power dissipated in the resistor can be substantially higher than the average power. However, since the voltage supplied to the resistor varies in time, so does the instantaneous power. Therefore, a better estimate of the energy dissipated in an AC circuit is given by the average power. For example, the power rating on light bulbs is in fact the average power dissipated in the bulb.

The RLC Circuits

A Series L-R-C Circuit: The Phasor Approach

**Learning Goal:** To understand the use of phasor diagrams in calculating the impedance and resonance conditions in a series L-R-C circuit.

In this problem, you will consider a series L-R-C circuit, containing a resistor of resistance \( R \), an inductor of inductance \( L \), and a capacitor of capacitance \( C \), all connected in series to an AC source providing an alternating voltage \( V(t) = V_0 \cos(\omega t) \).

You may have solved a number of problems in which you had to find the effective resistance of a circuit containing multiple resistors. Finding the overall resistance of a circuit is often of practical interest. In this problem, we will start our analysis of this L-R-C circuit by finding its effective overall resistance, or impedance. The impedance \( Z \) is defined by \( Z = \frac{V_0}{I_0} \) where \( V_0 \) and \( I_0 \) are the amplitudes of the voltage across the entire circuit and the current, respectively.

**Part A**

Find the impedance \( Z \) of the circuit using the phasor diagram shown. Notice that in this series circuit, the current is same for all elements of the circuit:

\[ Z = \text{Answer not displayed} \]

**Hints**

- Finding individual voltages
- Finding the overall voltage
- Combining the vectors

**Part B**

Now find the tangent of the phase angle \( \phi \) between the current and the overall voltage in this circuit.

\[ \tan(\phi) = \text{Answer not displayed} \]

**Part C**

Imagine that the parameters \( R, L, C \), and the amplitude of the voltage \( V_0 \) are fixed, but the frequency of the voltage source is changeable. If the frequency of the source is changed from a very low one to a very high one, the current amplitude \( I_0 \) will also change. The frequency at which \( I_0 \) is at a maximum is called resonance. Find the frequency \( \omega_0 \) at which the circuit reaches resonance.

**Hint**

Analyzing the impedance
Part D
What is the phase angle \( \phi \) between the voltage and the current when resonance is reached?

\[ \text{ANSWER: } \phi = \text{Answer not displayed} \]

Part E
Now imagine that the parameters \( R \), \( L \), \( \omega_0 \) and the amplitude of the voltage \( V_0 \) are fixed but that the value of \( C \) can be changed. This is one of the easiest parameters to change when "tuning" such (radio frequency) circuits in order to make them resonate. This is because the capacitance can be changed just by moving the capacitor plates closer or farther apart. Find the resonance value of capacitance \( C_\text{res} \).

Express your answer in terms of \( I \) and \( \omega_0 \).

\[ \text{ANSWER: } C_\text{res} = \text{Answer not displayed} \]

### Average Power in an L-R-C Circuit

A circuit consists of a resistor (resistance \( R \)), inductor (inductance \( L \)), and capacitor (capacitance \( C \)) connected in series with an AC source supplying sinusoidal voltage \( V_0 \cos(\omega t) \). Assume that all circuit elements are ideal, so that the only resistance in the circuit is due to the resistor. Also assume that \( \omega_0 \) is the resonant frequency of the circuit.

**Part A**
What is the average power \( P_\text{avg} \) supplied by the voltage source?

**Part A.1**
*Find the instantaneous power*

**Part A.2**
*Average value of a periodic function*

Express your answer in terms of any or all of the following quantities: \( V_0 \), \( C \), \( L \), and \( R \).

\[ \text{ANSWER: } P_\text{avg} = \text{Answer not displayed} \]

### Resonance in an R-L-C Circuit

In an \( R-L-C \) series circuit, the resistance is 420 ohms, the inductance is 0.380 henrys, and the capacitance is \( 1.00 \times 10^{-2} \) microfarads.

**Part A**
What is the resonance angular frequency \( \omega_0 \) of the circuit?

**Part A.1**
*Definition of the resonance angular frequency*

**Part A.2**
*Relationship between current and voltage amplitudes*

**Part A.3**
*What is an expression for impedance?*

**Part A.4**
*Finding the formula for the resonant frequency*

Express your answer in radians per second to three significant figures.

\[ \text{ANSWER: } \omega_0 = 1.62 \times 10^4 \text{ rad/s} \]

**Part B**
The capacitor can withstand a peak voltage of 540 volts. If the voltage source operates at the resonance frequency, what maximum voltage amplitude \( V_\text{max} \) can the source have if the maximum capacitor voltage is not exceeded?

**Part B.1**
*Voltage across a capacitor*

In a series \( L-R-C \) circuit the voltage across a capacitor is given by the equation
\[ V_C = I X_C, \]

where \( V_C \) is the voltage across the capacitor, \( I \) is the amplitude of the current through the capacitor, and \( X_C = \frac{1}{\omega C} \) is the capacitive reactance.

**Hint B.2**  
**Current at the resonance frequency**

Recall that at the resonance frequency the impedance \( Z \) is equal to the resistance \( R \). As a result, the current in the system is given by \( I = \frac{V_{\text{max}}}{R} \). Since we know the maximum voltage that the capacitor can handle, we should be able to combine this equation and the equation for capacitor voltage to determine a maximum source voltage \( V_{\text{max}} \).

Express your answer in volts to three significant figures.

**ANSWER:**

\[ V_{\text{max}} = 36.8 \ \text{V} \]