

9

Deflections of Beams

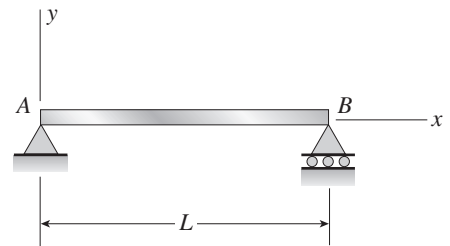
Differential Equations of the Deflection Curve

The beams described in the problems for Section 9.2 have constant flexural rigidity EI .

Problem 9.2-1 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x}{360EI}(7L^4 - 10L^2x^2 + 3x^4)$$

Describe the load acting on the beam.



Probs. 9.2-1 and 9.2-2

Solution 9.2-1 Simple beam

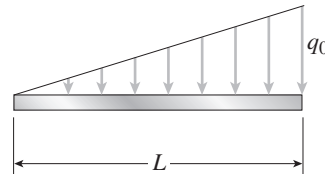
$$v = -\frac{q_0 x}{360EI}(7L^4 - 10L^2x^2 + 3x^4)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0 x}{LEI}$$

$$\text{From Eq. (9-12c): } q = -EIv'''' = \frac{q_0 x}{L} \quad \leftarrow$$

The load is a downward triangular load of maximum intensity q_0 . \leftarrow



Problem 9.2-2 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L}$$

- Describe the load acting on the beam.
- Determine the reactions R_A and R_B at the supports.
- Determine the maximum bending moment M_{\max} .

Solution 9.2-2 Simple beam

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L}$$

$$v' = -\frac{q_0 L^3}{\pi^3 EI} \cos \frac{\pi x}{L}$$

$$v'' = \frac{q_0 L^2}{\pi^2 EI} \sin \frac{\pi x}{L}$$

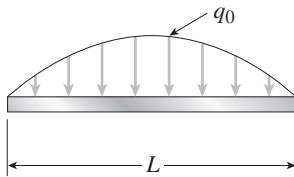
$$v''' = \frac{q_0 L}{\pi EI} \cos \frac{\pi x}{L}$$

$$v'''' = -\frac{q_0}{EI} \sin \frac{\pi x}{L}$$

(a) LOAD (EQ. 9-12c)

$$q = -EIv'''' = q_0 \sin \frac{\pi x}{L} \quad \leftarrow$$

The load has the shape of a sine curve, acts downward, and has maximum intensity q_0 . \leftarrow



(b) REACTIONS (EQ. 9-12b)

$$V = EIv''' = -\frac{q_0 L}{\pi} \cos \frac{\pi x}{L}$$

At $x = 0$: $V = R_A = -\frac{q_0 L}{\pi} \quad \leftarrow$

At $x = L$: $V = -R_B = -\frac{q_0 L}{\pi}$; $R_B = \frac{q_0 L}{\pi} \quad \leftarrow$

(c) MAXIMUM BENDING MOMENT (EQ. 9-12a)

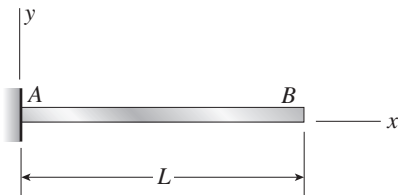
$$M = EIv'' = \frac{q_0 L^2}{\pi^2} \sin \frac{\pi x}{L}$$

For maximum moment, $x = \frac{L}{2}$; $M_{\max} = \frac{q_0 L^2}{\pi^2} \quad \leftarrow$

Problem 9.2-3 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

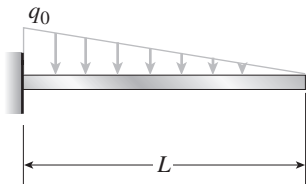
$$v = -\frac{q_0 x^2}{120 LEI} (10L^3 - 10L^2x + 5Lx^2 - x^3)$$

Describe the load acting on the beam.



Probs. 9.2-3 and 9.2-4

Solution 9.2-3 Cantilever beam



$$v = -\frac{q_0 x^2}{120 LEI} (10L^3 - 10L^2x + 5Lx^2 - x^3)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0}{LEI} (L - x)$$

From Eq. (9-12c):

$$q = -EIv'''' = q_0 \left(1 - \frac{x}{L}\right) \quad \leftarrow$$

The load is a downward triangular load of maximum intensity q_0 . \leftarrow

Problem 9.2-4 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x^2}{360 L^2 EI} (45L^4 - 40L^3 x + 15L^2 x^2 - x^4)$$

- (a) Describe the load acting on the beam.
 (b) Determine the reactions R_A and M_A at the support.

Solution 9.2-4 Cantilever beam

$$v = -\frac{q_0 x^2}{360 L^2 EI} (45L^4 - 40L^3 x + 15L^2 x^2 - x^4)$$

$$v' = -\frac{q_0}{60 L^2 EI} (15L^4 x - 20L^3 x^2 + 10L^2 x^3 - x^5)$$

$$v'' = -\frac{q_0}{12 L^2 EI} (3L^4 - 8L^3 x + 6L^2 x^2 - x^4)$$

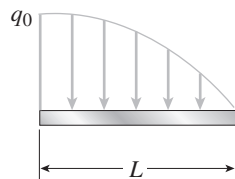
$$v''' = -\frac{q_0}{3 L^2 EI} (-2L^3 + 3L^2 x - x^3)$$

$$v'''' = -\frac{q_0}{L^2 EI} (L^2 - x^2)$$

(a) LOAD (EQ. 9-12c)

$$q = -EIv'''' = q_0 \left(1 - \frac{x^2}{L^2}\right) \quad \leftarrow$$

The load is a downward parabolic load of maximum intensity q_0 . \leftarrow



(b) REACTIONS R_A AND M_A (EQ. 9-12b AND EQ. 9-12a)

$$V = EIv''' = -\frac{q_0}{3L^2} (-2L^3 + 3L^2 x - x^3)$$

$$\text{At } x = 0: \quad V = R_A = \frac{2q_0 L}{3} \quad \leftarrow$$

$$M = EIv'' = -\frac{q_0}{12L^2} (3L^4 - 8L^3 x + 6L^2 x^2 - x^4)$$

$$\text{At } x = 0: \quad M = M_A = -\frac{q_0 L^2}{4} \quad \leftarrow$$

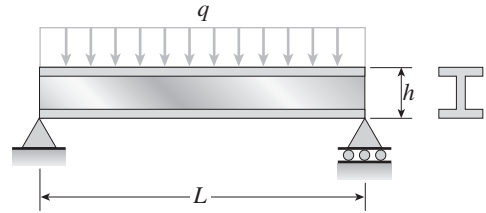
NOTE: Reaction R_A is positive upward.
 Reaction M_A is positive clockwise (minus means M_A is counterclockwise).

Deflection Formulas

Problems 9.3-1 through 9.3-7 require the calculation of deflections using the formulas derived in Examples 9-1, 9-2, and 9-3. All beams have constant flexural rigidity EI .

Problem 9.3-1 A wide-flange beam (W 12 × 35) supports a uniform load on a simple span of length $L = 14$ ft (see figure).

Calculate the maximum deflection δ_{\max} at the midpoint and the angles of rotation θ at the supports if $q = 1.8$ k/ft and $E = 30 \times 10^6$ psi. Use the formulas of Example 9-1.



Probs. 9.3-1, 9.3-2 and 9.3-3

Solution 9.3-1 Simple beam (uniform load)

$$\begin{aligned} W 12 \times 35 \quad L &= 14 \text{ ft} = 168 \text{ in.} \\ q &= 1.8 \text{ k/ft} = 150 \text{ lb/in.} \quad E = 30 \times 10^6 \text{ psi} \\ I &= 285 \text{ in.}^4 \end{aligned}$$

MAXIMUM DEFLECTION (EQ. 9-18)

$$\begin{aligned} \delta_{\max} &= \frac{5qL^4}{384EI} = \frac{5(150 \text{ lb/in.})(168 \text{ in.})^4}{384(30 \times 10^6 \text{ psi})(285 \text{ in.}^4)} \\ &= 0.182 \text{ in.} \quad \leftarrow \end{aligned}$$

ANGLE OF ROTATION AT THE SUPPORTS

(EQS. 9-19 AND 9-20)

$$\begin{aligned} \theta &= \theta_A = \theta_B = \frac{qL^3}{24EI} = \frac{(150 \text{ lb/in.})(168 \text{ in.})^3}{24(30 \times 10^6 \text{ psi})(285 \text{ in.}^4)} \\ &= 0.003466 \text{ rad} = 0.199^\circ \quad \leftarrow \end{aligned}$$

Problem 9.3-2 A uniformly loaded steel wide-flange beam with simple supports (see figure) has a downward deflection of 10 mm at the midpoint and angles of rotation equal to 0.01 radians at the ends.

Calculate the height h of the beam if the maximum bending stress is 90 MPa and the modulus of elasticity is 200 GPa. (Hint: Use the formulas of Example 9-1.)

Solution 9.3-2 Simple beam (uniform load)

$$\begin{aligned} \delta &= \delta_{\max} = 10 \text{ mm} \quad \theta = \theta_A = \theta_B = 0.01 \text{ rad} \\ \sigma &= \sigma_{\max} = 90 \text{ MPa} \quad E = 200 \text{ GPa} \end{aligned}$$

Calculate the height h of the beam.

$$\text{Eq. (9-18): } \delta = \delta_{\max} = \frac{5qL^4}{384EI} \text{ or } q = \frac{384EI\delta}{5L^4} \quad (1)$$

$$\text{Eq. (9-19): } \theta = \theta_A = \frac{qL^3}{24EI} \text{ or } q = \frac{24EI\theta}{L^3} \quad (2)$$

$$\text{Equate (1) and (2) and solve for } L: L = \frac{16\delta}{5\theta} \quad (3)$$

$$\text{Flexure formula: } \sigma = \frac{Mc}{I} = \frac{Mh}{2I}$$

Maximum bending moment:

$$M = \frac{qL^2}{8} \quad \therefore \sigma = \frac{qL^2h}{16I} \quad (4)$$

$$\text{Solve Eq. (4) for } h: h = \frac{16I\sigma}{qL^2} \quad (5)$$

Substitute for q from (2) and for L from (3):

$$h = \frac{32\sigma\delta}{15E\theta^2} \quad \leftarrow$$

Substitute numerical values:

$$h = \frac{32(90 \text{ MPa})(10 \text{ mm})}{15(200 \text{ GPa})(0.01 \text{ rad})^2} = 96 \text{ mm} \quad \leftarrow$$

Problem 9.3-3 What is the span length L of a uniformly loaded simple beam of wide-flange cross section (see figure) if the maximum bending stress is 12,000 psi, the maximum deflection is 0.1 in., the height of the beam is 12 in., and the modulus of elasticity is 30×10^6 psi? (Use the formulas of Example 9-1.)

Solution 9.3-3 Simple beam (uniform load)

$$\sigma = \sigma_{\max} = 12,000 \text{ psi} \quad \delta = \delta_{\max} = 0.1 \text{ in.}$$

$$h = 12 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$$

Calculate the span length L .

$$\text{Eq. (9-18): } \delta = \delta_{\max} = \frac{5qL^4}{384EI} \text{ or } q = \frac{384EI\delta}{5L^4} \quad (1)$$

$$\text{Flexure formula: } \sigma = \frac{Mc}{I} = \frac{Mh}{2I}$$

Maximum bending moment:

$$M = \frac{qL^2}{8} \quad \therefore \sigma = \frac{qL^2h}{16I} \quad (2)$$

$$\text{Solve Eq. (2) for } q: \quad q = \frac{16I\sigma}{L^2h} \quad (3)$$

Equate (1) and (2) and solve for L :

$$L^2 = \frac{24Eh\delta}{5\sigma} \quad L = \sqrt{\frac{24Eh\delta}{5\sigma}} \quad \leftarrow$$

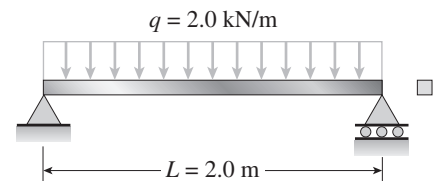
Substitute numerical values:

$$L^2 = \frac{24(30 \times 10^6 \text{ psi})(12 \text{ in.})(0.1 \text{ in.})}{5(12,000 \text{ psi})} = 14,400 \text{ in.}^2$$

$$L = 120 \text{ in.} = 10 \text{ ft} \quad \leftarrow$$

Problem 9.3-4 Calculate the maximum deflection δ_{\max} of a uniformly loaded simple beam (see figure) if the span length $L = 2.0$ m, the intensity of the uniform load $q = 2.0$ kN/m, and the maximum bending stress $\sigma = 60$ MPa.

The cross section of the beam is square, and the material is aluminum having modulus of elasticity $E = 70$ GPa. (Use the formulas of Example 9-1.)



Solution 9.3-4 Simple beam (uniform load)

$$L = 2.0 \text{ m} \quad q = 2.0 \text{ kN/m}$$

$$\sigma = \sigma_{\max} = 60 \text{ MPa} \quad E = 70 \text{ GPa}$$

CROSS SECTION (square; $b =$ width)

$$I = \frac{b^4}{12} \quad S = \frac{b^3}{6}$$

$$\text{Maximum deflection (Eq. 9-18): } \delta = \frac{5qL^4}{384EI} \quad (1)$$

$$\text{Substitute for } I: \delta = \frac{5qL^4}{32Eb^4} \quad (2)$$

$$\text{Flexure formula with } M = \frac{qL^2}{8}: \quad \sigma = \frac{M}{S} = \frac{qL^2}{8S}$$

$$\text{Substitute for } S: \sigma = \frac{3qL^2}{4b^3} \quad (3)$$

$$\text{Solve for } b^3: b^3 = \frac{3qL^2}{4\sigma} \quad (4)$$

$$\text{Substitute } b \text{ into Eq. (2): } \delta_{\max} = \frac{5L\sigma}{24E} \left(\frac{4L\sigma}{3q} \right)^{1/3} \quad \leftarrow$$

(The term in parentheses is nondimensional.)

Substitute numerical values:

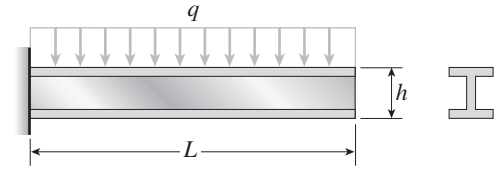
$$\frac{5L\sigma}{24E} = \frac{5(2.0 \text{ m})(60 \text{ MPa})}{24(70 \text{ GPa})} = \frac{1}{2800} \text{ m} = \frac{1}{2.8} \text{ mm}$$

$$\left(\frac{4L\sigma}{3q} \right)^{1/3} = \left[\frac{4(2.0 \text{ m})(60 \text{ MPa})}{3(2000 \text{ N/m})} \right]^{1/3} = 10(80)^{1/3}$$

$$\delta_{\max} = \frac{10(80)^{1/3}}{2.8} \text{ mm} = 15.4 \text{ mm} \quad \leftarrow$$

Problem 9.3-5 A cantilever beam with a uniform load (see figure) has a height h equal to $1/8$ of the length L . The beam is a steel wide-flange section with $E = 28 \times 10^6$ psi and an allowable bending stress of 17,500 psi in both tension and compression.

Calculate the ratio δ/L of the deflection at the free end to the length, assuming that the beam carries the maximum allowable load. (Use the formulas of Example 9-2.)



Solution 9.3-5 Cantilever beam (uniform load)

$$\frac{h}{L} = \frac{1}{8} \quad E = 28 \times 10^6 \text{ psi} \quad \sigma = 17,500 \text{ psi}$$

Calculate the ratio δ/L .

$$\text{Maximum deflection (Eq. 9-26): } \delta_{\max} = \frac{qL^4}{8EI} \quad (1)$$

$$\therefore \frac{\delta}{L} = \frac{qL^3}{8EI} \quad (2)$$

Flexure formula with $M = \frac{qL^2}{2}$:

$$\sigma = \frac{Mc}{I} = \left(\frac{qL^2}{2}\right)\left(\frac{h}{2I}\right) = \frac{qL^2h}{4I}$$

Solve for q :

$$q = \frac{4I\sigma}{L^2h} \quad (3)$$

Substitute q from (3) into (2):

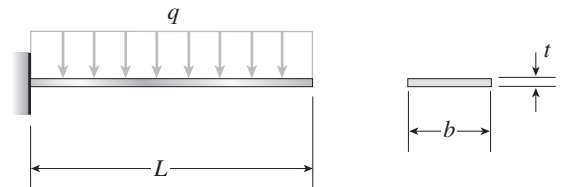
$$\frac{\delta}{L} = \frac{\sigma}{2E} \left(\frac{L}{h}\right) \quad \leftarrow$$

Substitute numerical values:

$$\frac{\delta}{L} = \frac{17,500 \text{ psi}}{2(28 \times 10^6 \text{ psi})} (8) = \frac{1}{400} \quad \leftarrow$$

Problem 9.3-6 A gold-alloy microbeam attached to a silicon wafer behaves like a cantilever beam subjected to a uniform load (see figure). The beam has length $L = 27.5 \mu\text{m}$ and rectangular cross section of width $b = 4.0 \mu\text{m}$ and thickness $t = 0.88 \mu\text{m}$. The total load on the beam is $17.2 \mu\text{N}$.

If the deflection at the end of the beam is $2.46 \mu\text{m}$, what is the modulus of elasticity E_g of the gold alloy? (Use the formulas of Example 9-2.)



Solution 9.3-6 Gold-alloy microbeam

Cantilever beam with a uniform load.

$$L = 27.5 \mu\text{m} \quad b = 4.0 \mu\text{m} \quad t = 0.88 \mu\text{m}$$

$$qL = 17.2 \mu\text{N} \quad \delta_{\max} = 2.46 \mu\text{m}$$

Determine Eq.

$$\text{Eq. (9-26): } \delta = \frac{qL^4}{8E_q I} \quad \text{or} \quad E_q = \frac{qL^4}{8I\delta_{\max}}$$

$$I = \frac{bt^3}{12} \quad E_q = \frac{3qL^4}{2bt^3\delta_{\max}} \quad \leftarrow$$

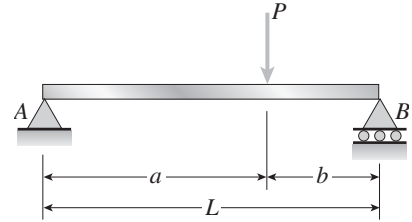
Substitute numerical values:

$$E_q = \frac{3(17.2 \mu\text{N})(27.5 \mu\text{m})^3}{2(4.0 \mu\text{m})(0.88 \mu\text{m})^3(2.46 \mu\text{m})}$$

$$= 80.02 \times 10^9 \text{ N/m}^2 \quad \text{or} \quad E_q = 80.0 \text{ GPa} \quad \leftarrow$$

Problem 9.3-7 Obtain a formula for the ratio δ_c/δ_{\max} of the deflection at the midpoint to the maximum deflection for a simple beam supporting a concentrated load P (see figure).

From the formula, plot a graph of δ_c/δ_{\max} versus the ratio a/L that defines the position of the load ($0.5 < a/L < 1$). What conclusion do you draw from the graph? (Use the formulas of Example 9-3.)



Solution 9.3-7 Simple beam (concentrated load)

$$\text{Eq. (9-35): } \delta_c = \frac{Pb(3L^2 - 4b^2)}{48EI} \quad (a \geq b)$$

$$\text{Eq. (9-34): } \delta_{\max} = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}LEI} \quad (a \geq b)$$

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3}L)(3L^2 - 4b^2)}{16(L^2 - b^2)^{3/2}} \quad (a \geq b)$$

Replace the distance b by the distance a by substituting $L - a$ for b :

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3}L)(-L^2 + 8ab - 4a^2)}{16(2aL - a^2)^{3/2}}$$

Divide numerator and denominator by L^2 :

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16L\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}}$$

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3})\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}} \quad \leftarrow$$

ALTERNATIVE FORM OF THE RATIO

$$\text{Let } \beta = \frac{a}{L}$$

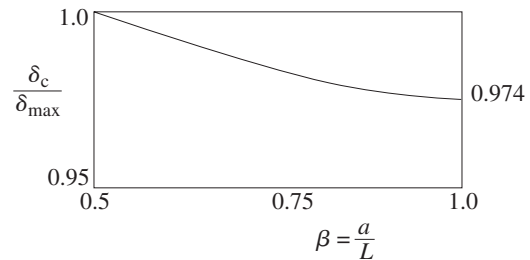
$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3})(-1 + 8\beta - 4\beta^2)}{16(2\beta - \beta^2)^{3/2}} \quad \leftarrow$$

GRAPH OF δ_c/δ_{\max} VERSUS $\beta = a/L$

Because $a \geq b$, the ratio β varies from 0.5 to 1.0.

β	$\frac{\delta_c}{\delta_{\max}}$
0.5	1.0
0.6	0.996
0.7	0.988
0.8	0.981
0.9	0.976
1.0	0.974

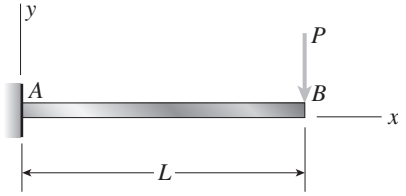
NOTE: The deflection δ_c at the midpoint of the beam is almost as large as the maximum deflection δ_{\max} . The greatest difference is only 2.6% and occurs when the load reaches the end of the beam ($\beta = 1$).



Deflections by Integration of the Bending-Moment Equation

Problems 9.3-8 through 9.3-16 are to be solved by integrating the second-order differential equation of the deflection curve (the bending-moment equation). The origin of coordinates is at the left-hand end of each beam, and all beams have constant flexural rigidity EI .

Problem 9.3-8 Derive the equation of the deflection curve for a cantilever beam AB supporting a load P at the free end (see figure). Also, determine the deflection δ_B and angle of rotation θ_B at the free end. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-8 Cantilever beam (concentrated load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -P(L - x)$$

$$EIv' = -PLx + \frac{Px^2}{2} + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -\frac{PLx^2}{2} + \frac{Px^3}{6} + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

$$v = -\frac{Px^2}{6EI}(3L - x) \quad \leftarrow$$

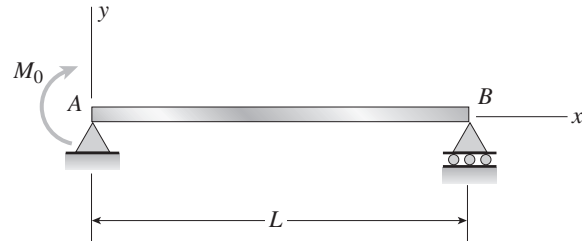
$$v' = -\frac{Px}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{PL^3}{3EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{PL^2}{2EI} \quad \leftarrow$$

(These results agree with Case 4, Table G-1.)

Problem 9.3-9 Derive the equation of the deflection curve for a simple beam AB loaded by a couple M_0 at the left-hand support (see figure). Also, determine the maximum deflection δ_{\max} . (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-9 Simple beam (couple M_0)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = M_0\left(1 - \frac{x}{L}\right)$$

$$EIv' = M_0\left(x - \frac{x^2}{2L}\right) + C_1$$

$$EIv = M_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1x + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

B.C. $v(L) = 0 \quad \therefore C_1 = -\frac{M_0L}{3}$

$$v = -\frac{M_0x}{6LEI}(2L^2 - 3Lx + x^2) \quad \leftarrow$$

MAXIMUM DEFLECTION

$$v' = -\frac{M_0}{6LEI}(2L^2 - 6Lx + 3x^2)$$

Set $v' = 0$ and solve for x :

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right) \quad \leftarrow$$

Substitute x_1 into the equation for v :

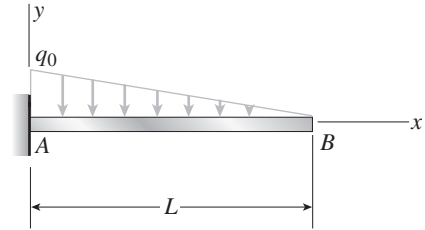
$$\delta_{\max} = -(v)_{x=x_1}$$

$$= \frac{M_0L^2}{9\sqrt{3}EI} \quad \leftarrow$$

(These results agree with Case 7, Table G-2.)

Problem 9.3-10 A cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0 is shown in the figure.

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-10 Cantilever beam (triangular load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q_0}{6L}(L-x)^3$$

$$EIv' = \frac{q_0}{24L}(L-x)^4 + C_1$$

$$\text{B.C. } v'(0) = 0 \quad \therefore C_1 = -\frac{q_0 L^3}{24}$$

$$EIv = -\frac{q_0}{120L}(L-x)^5 - \frac{q_0 L^3 x}{24} + C_2$$

$$\text{B.C. } v(0) = 0 \quad \therefore C_2 = \frac{q_0 L^4}{120}$$

$$v = -\frac{q_0 x^2}{120 LEI} (10L^3 - 10L^2x + 5Lx^2 - x^3) \quad \leftarrow$$

$$v' = -\frac{q_0 x}{24 LEI} (4L^3 - 6L^2x + 4Lx^2 - x^3)$$

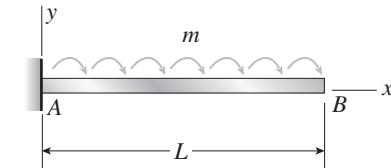
$$\delta_B = -v(L) = \frac{q_0 L^4}{30 EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{24 EI} \quad \leftarrow$$

(These results agree with Case 8, Table G-1.)

Problem 9.3-11 A cantilever beam AB is acted upon by a uniformly distributed moment (bending moment, not torque) of intensity m per unit distance along the axis of the beam (see figure).

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-11 Cantilever beam (distributed moment)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -m(L-x)$$

$$EIv' = -m\left(Lx - \frac{x^2}{2}\right) + C_1$$

$$\text{B.C. } v'(0) = 0 \quad \therefore C_1 = 0$$

$$EIv = -m\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

$$\text{B.C. } v(0) = 0 \quad \therefore C_2 = 0$$

$$v = -\frac{mx^2}{6 EI} (3L-x) \quad \leftarrow$$

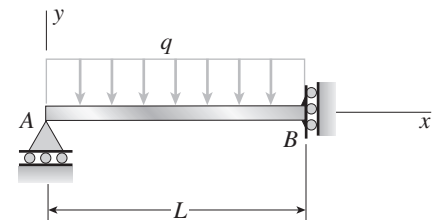
$$v' = -\frac{mx}{2 EI} (2L-x)$$

$$\delta_B = -v(L) = \frac{mL^3}{3 EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{mL^2}{2 EI} \quad \leftarrow$$

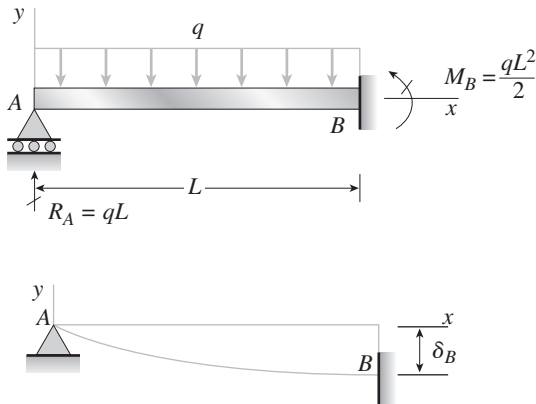
Problem 9.3-12 The beam shown in the figure has a roller support at A and a guided support at B . The guided support permits vertical movement but no rotation.

Derive the equation of the deflection curve and determine the deflection δ_B at end B due to the uniform load of intensity q . (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-12 Beam with a guided support

REACTIONS AND DEFLECTION CURVE



BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = qLx - \frac{qx^2}{2}$$

$$EIv' = \frac{qLx^2}{2} - \frac{qx^3}{6} + C_1$$

$$\text{B.C. } v(L) = 0 \quad \therefore C_1 = -\frac{qL^3}{3}$$

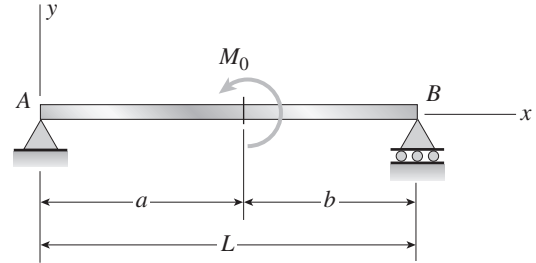
$$EIv = \frac{qLx^3}{6} - \frac{qx^4}{24} - \frac{qL^3x}{3} + C_2$$

$$\text{B.C. } v(0) = 0 \quad \therefore C_2 = 0$$

$$v = -\frac{qx}{24EI}(8L^3 - 4Lx^2 + x^3) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{5qL^4}{24EI} \quad \leftarrow$$

Problem 9.3-13 Derive the equations of the deflection curve for a simple beam AB loaded by a couple M_0 acting at distance a from the left-hand support (see figure). Also, determine the deflection δ_0 at the point where the load is applied. (Note: Use the second-order differential equation of the deflection curve.)

**Solution 9.3-13** Simple beam (couple M_0)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = \frac{M_0x}{L} \quad (0 \leq x \leq a)$$

$$EIv' = \frac{M_0x^2}{2L} + C_1 \quad (0 \leq x \leq a)$$

$$EIv'' = M = -\frac{M_0}{L}(L-x) \quad (a \leq x \leq L)$$

$$EIv' = -\frac{M_0}{L}\left(Lx - \frac{x^2}{2}\right) + C_2 \quad (a \leq x \leq L)$$

$$\text{B.C. 1 } (v')_{\text{Left}} = (v')_{\text{Right}} \quad \text{at } x = a$$

$$\therefore C_2 = C_1 + M_0a$$

$$EIv = \frac{M_0x^3}{6L} + C_1x + C_3 \quad (0 \leq x \leq a)$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_3 = 0$$

$$EIv = -\frac{M_0x^2}{2} + \frac{M_0x^3}{6L} + C_1x + M_0ax + C_4$$

$$(a \leq x \leq L)$$

$$\text{B.C. 3 } v(L) = 0 \quad \therefore C_4 = -M_0L\left(a - \frac{L}{3}\right) - C_1L$$

$$\text{B.C. 4 } (v)_{\text{Left}} = (v)_{\text{Right}} \quad \text{at } x = a$$

$$\therefore C_4 = -\frac{M_0a^2}{2}$$

$$C_1 = \frac{M_0}{6L}(2L^2 - 6aL + 3a^2)$$

$$v = -\frac{M_0x}{6LEI}(6aL - 3a^2 - 2L^2 - x^2) \quad (0 \leq x \leq a) \quad \leftarrow$$

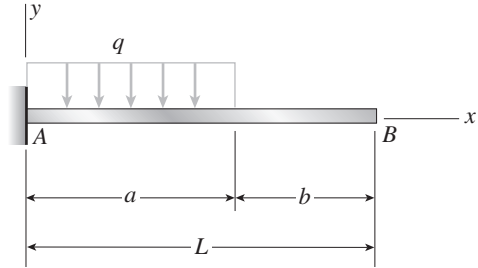
$$v = -\frac{M_0}{6LEI}(3a^2L - 3a^2x - 2L^2x + 3Lx^2 - x^3) \quad (a \leq x \leq L) \quad \leftarrow$$

$$\delta_0 = -v(a) = \frac{M_0a(L-a)(2a-L)}{3LEI}$$

$$= \frac{M_0ab(2a-L)}{3LEI} \quad \leftarrow$$

NOTE: δ_0 is positive downward. The pending results agree with Case 9, Table G-2.

Problem 9.3-14 Derive the equations of the deflection curve for a cantilever beam AB carrying a uniform load of intensity q over part of the span (see figure). Also, determine the deflection δ_B at the end of the beam. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-14 Cantilever beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q}{2}(a-x)^2 = -\frac{q}{2}(a^2 - 2ax + x^2) \quad (0 \leq x \leq a)$$

$$EIv' = -\frac{q}{2}\left(a^2x - ax^2 + \frac{x^3}{3}\right) + C_1 \quad (0 \leq x \leq a)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv'' = M = 0 \quad (a \leq x \leq L)$$

$$EIv' = C_2 \quad (a \leq x \leq L)$$

B.C. 2 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a$

$$\therefore C_2 = -\frac{qa^3}{6}$$

$$EIv = -\frac{q}{2}\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + C_3 \quad (0 \leq x \leq a)$$

B.C. 3 $v(0) = 0 \quad \therefore C_3 = 0$

$$EIv = C_2x + C_4 = -\frac{qa^3x}{6} + C_4 \quad (a \leq x \leq L)$$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a$

$$\therefore C_4 = \frac{qa^4}{24}$$

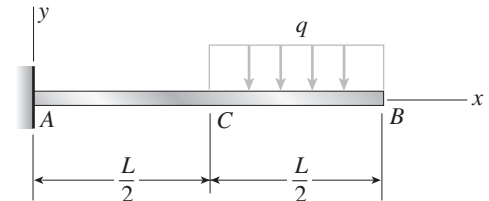
$$v = -\frac{qx^2}{24EI}(6a^2 - 4ax + x^2) \quad (0 \leq x \leq a) \quad \leftarrow$$

$$v = -\frac{qa^3}{24EI}(4x - a) \quad (a \leq x \leq L) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{qa^3}{24EI}(4L - a) \quad \leftarrow$$

(These results agree with Case 2, Table G-1.)

Problem 9.3-15 Derive the equations of the deflection curve for a cantilever beam AB supporting a uniform load of intensity q acting over one-half of the length (see figure). Also, obtain formulas for the deflections δ_B and δ_C at points B and C , respectively. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-15 Cantilever beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{qL}{8}(3L - 4x) \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$EIv' = -\frac{qL}{8}(3Lx - 2x^2) + C_1 \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv'' = M = -\frac{q}{2}(L^2 - 2Lx + x^2) \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$EIv' = -\frac{q}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) + C_2 \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$\text{B.C. 2 } (v')_{\text{Left}} = (v')_{\text{Right}} \text{ at } x = \frac{L}{2}$$

$$\therefore C_2 = \frac{qL^3}{48}$$

$$EIv = -\frac{qL}{8} \left(\frac{3Lx^2}{2} - \frac{2x^3}{3} \right) + C_3 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\text{B.C. 3 } v(0) = 0 \quad \therefore C_3 = 0$$

$$EIv = -\frac{q}{2} \left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12} \right) + \frac{qL^3}{48}x + C_4$$

$$\left(\frac{L}{2} \leq x \leq L \right)$$

$$\text{B.C. 4 } (v)_{\text{Left}} = (v)_{\text{Right}} \text{ at } x = \frac{L}{2}$$

$$\therefore C_4 = -\frac{qL^4}{384}$$

$$v = -\frac{qLx^2}{48EI} (9L - 4x) \quad \left(0 \leq x \leq \frac{L}{2} \right) \quad \leftarrow$$

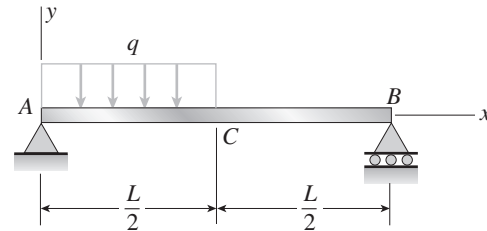
$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{7qL^4}{192EI} \quad \leftarrow$$

$$v = -\frac{q}{384EI} (16x^4 - 64Lx^3 + 96L^2x^2 - 8L^3x + L^4)$$

$$\left(\frac{L}{2} \leq x \leq L \right) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{41qL^4}{384EI} \quad \leftarrow$$

Problem 9.3-16 Derive the equations of the deflection curve for a simple beam AB with a uniform load of intensity q acting over the left-hand half of the span (see figure). Also, determine the deflection δ_C at the midpoint of the beam. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-16 Simple beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = \frac{3qLx}{8} - \frac{qx^2}{2} \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$EIv' = \frac{3qLx^2}{16} - \frac{qx^3}{6} + C_1 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$EIv'' = M = \frac{qL^2}{8} - \frac{qLx}{8} \quad \left(\frac{L}{2} \leq x \leq L \right)$$

$$EIv' = \frac{qL^2x}{8} - \frac{qLx^2}{16} + C_2 \quad \left(\frac{L}{2} \leq x \leq L \right)$$

$$\text{B.C. 1 } (v')_{\text{Left}} = (v')_{\text{Right}} \text{ at } x = \frac{L}{2}$$

$$\therefore C_2 = C_1 - \frac{qL^2}{48}$$

$$EIv = \frac{qLx^3}{16} - \frac{qx^4}{24} + C_1x + C_3 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_3 = 0$$

$$EIv = \frac{qL^2x^2}{16} - \frac{qLx^3}{48} + C_1x - \frac{qL^3x}{48} + C_4$$

$$\left(\frac{L}{2} \leq x \leq L \right)$$

$$\text{B.C. 3 } v(L) = 0 \quad \therefore C_4 = -C_1L - \frac{qL^4}{48}$$

$$\text{B.C. 4 } (v)_{\text{Left}} = (v)_{\text{Right}} \text{ at } x = \frac{L}{2}$$

$$\therefore C_1 = -\frac{3qL^3}{128}$$

$$v = -\frac{qx}{384EI} (9L^3 - 24Lx^2 + 16x^3) \quad \left(0 \leq x \leq \frac{L}{2} \right) \quad \leftarrow$$

$$v = -\frac{qL}{384EI} (8x^3 - 24Lx^2 + 17L^2x - L^3) \quad \leftarrow$$

$$\left(\frac{L}{2} \leq x \leq L \right)$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{5qL^4}{768EI} \quad \leftarrow$$

(These results agree with Case 2, Table G-2.)