Outline

Extension principle
Fuzzy relations
Fuzzy if-then rules
Compositional rule of inference
Fuzzy reasoning
Extension Principle

A is a fuzzy set on $X$:

$$A = \mu_A(x_1) / x_1 + \mu_A(x_2) / x_2 + \cdots + \mu_A(x_n) / x_n$$

The image of $A$ under $f()$ is a fuzzy set $B$:

$$B = \mu_B(y_1) / y_1 + \mu_B(y_2) / y_2 + \cdots + \mu_B(y_n) / y_n$$

where $y_i = f(x_i), i = 1$ to $n$.

If $f()$ is a many-to-one mapping, then

$$\mu_B(y) = \max_{x = f^{-1}(y)} \mu_A(x)$$
Fuzzy Relations

A fuzzy relation $R$ is a 2D MF:

$$R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}$$

Examples:

- $x$ is close to $y$ ($x$ and $y$ are numbers)
- $x$ depends on $y$ ($x$ and $y$ are events)
- $x$ and $y$ look alike ($x$, and $y$ are persons or objects)
- If $x$ is large, then $y$ is small ($x$ is an observed reading and $y$ is a corresponding action)
Max-Min Composition

The max-min composition of two fuzzy relations $R_1$ (defined on $X$ and $Y$) and $R_2$ (defined on $Y$ and $Z$) is

$$\mu_{R_1 \circ R_2}(x, z) = \bigvee_y \left[ \mu_{R_1}(x, y) \land \mu_{R_2}(y, z) \right]$$

Properties:

- **Associativity:**
  $$R \circ (S \circ T) = (R \circ S) \circ T$$

- **Distributivity over union:**
  $$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$$

- **Weak distributivity over intersection:**
  $$R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T)$$

- **Monotonicity:**
  $$S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$$
Max-Star Composition

Max-product composition:

\[ \mu_{R_1 \circ R_2}(x, z) = \bigvee_y [\mu_{R_1}(x, y) \mu_{R_2}(y, z)] \]

In general, we have max-* composition:

\[ \mu_{R_1 \circ R_2}(x, z) = \bigvee_y [\mu_{R_1}(x, y) \ast \mu_{R_2}(y, z)] \]

where * is a T-norm operator.
Linguistic Variables

A numerical variables takes numerical values:

\[ Age = 65 \]

A linguistic variables takes linguistic values:

\[ Age \text{ is old} \]

A linguistic values is a fuzzy set.

All linguistic values form a term set:

\[ T(\text{age}) = \{\text{young, not young, very young, ...}\]

\[ \text{middle aged, not middle aged, ...}\]

\[ \text{old, not old, very old, more or less old, ...}\]

\[ \text{not very yound and not very old, ...}\} \]
Linguistic Values (Terms)

(a) Primary Linguistic Values

(b) Composite Linguistic Values

\( X = \text{age} \)
Operations on Linguistic Values

Concentration: \(\text{CON}(A) = A^2\)

Dilation: \(\text{DIL}(A) = A^{0.5}\)

Contrast

\[\text{INT}(A) = \begin{cases} 2A^2, & 0 \leq \mu_A(x) \leq 0.5 \\ -2(-A)^2, & 0.5 \leq \mu_A(x) \leq 1 \end{cases}\]

Effects of Contrast Intensifier

intensif.m
Fuzzy If-Then Rules

General format:
If $x$ is $A$ then $y$ is $B$

Examples:
- If pressure is high, then volume is small.
- If the road is slippery, then driving is dangerous.
- If a tomato is red, then it is ripe.
- If the speed is high, then apply the brake a little.
Fuzzy If-Then Rules

Two ways to interpret “If x is A then y is B”:

A coupled with B

A entails B
Fuzzy Rules and Fuzzy Reasoning

Fuzzy If-Then Rules

Two ways to interpret “If x is A then y is B”:

- **A coupled with B:** \((A \text{ and } B)\)
  
  \[ R = A \to B = A \times B = \int \mu_A(x) \cdot \mu_B(y) |(x, y) \]

- **A entails B:** \((\text{not } A \text{ or } B)\)
  
  - Material implication
  - Propositional calculus
  - Extended propositional calculus
  - Generalization of modus ponens
Fuzzy If-Then Rules

Fuzzy implication function:

\[ \mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b) \]

A coupled with B
Fuzzy Rules and Fuzzy Reasoning

Fuzzy If-Then Rules

A entails B

(a) Zadeh's Arithmetic Rule  (b) Zadeh's Max-Min Rule  
(c) Boolean Fuzzy Implication (d) Goguen's Fuzzy Implication

fuzimp.m
Compositional Rule of Inference

Derivation of $y = b$ from $x = a$ and $y = f(x)$:

- $a$ and $b$: points
- $y = f(x)$: a curve

- $a$ and $b$: intervals
- $y = f(x)$: an interval-valued function
Compositional Rule of Inference

\( a \) is a fuzzy set and \( y = f(x) \) is a fuzzy relation:
Fuzzy Reasoning

Single rule with single antecedent

Rule: if x is A then y is B
Fact: x is A'
Conclusion: y is B'

Graphic Representation:
Fuzzy Reasoning

Single rule with multiple antecedent
Rule: if x is A and y is B then z is C
Fact: x is A' and y is B'
Conclusion: z is C'

Graphic Representation:
Fuzzy Reasoning

\[ C' = (A' \times B') \circ (A \times B \rightarrow C) \]

Premise 1

\[ \mu_{C'}(z) = \bigvee_{x,y} \left[ \mu_{A'}(x) \land \mu_{B'}(y) \right] \land \left[ \mu_A(x) \land \mu_B(y) \land \mu_C(z) \right] \]

Premise 2

\[ = \bigvee_{x,y} \left\{ \mu_{A'}(x) \land \mu_{B'}(y) \land \mu_A(x) \land \mu_B(y) \right\} \land \mu_C(z) \]

\[ = \left\{ \bigvee_x \left[ \mu_{A'}(x) \land \mu_A(x) \right] \right\} \land \left\{ \bigvee_y \left[ \mu_{B'}(y) \land \mu_B(y) \right] \right\} \land \mu_C(z) \]

\[ = (w_1 \land w_2) \land \mu_C(z) \]
Multiple rules with multiple antecedent

Rule 1: if x is $A_1$ and y is $B_1$ then z is $C_1$
Rule 2: if x is $A_2$ and y is $B_2$ then z is $C_2$
Fact: x is $A'$ and y is $B'$
Conclusion: z is $C'$

Graphic Representation: (next slide)
Fuzzy Reasoning

Graphics representation:

- $A'$, $A_1$ and $B'$, $B_1$ in the $X$ and $Y$ axes, respectively.
- $C_1$ and $C_2$ in the $Z$ axis.
- $x$ is $A'$, $y$ is $B'$, and $z$ is $C'$.

T-norm and weighted values $w_1$ and $w_2$. 

Fuzzy Rules and Fuzzy Reasoning
Fuzzy Reasoning: MATLAB Demo

>> ruleview mam21
Other Variants

Some terminology:

- Degrees of compatibility (match)
- Firing strength
- Qualified (induced) MFs
- Overall output MF