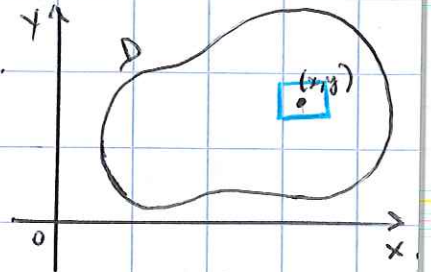
13.6 Applications of double integrals.

Idea: give the formulas to compute for example the center of mass of a lamina with variable density $\rho(x,y)$ seen as a function of 2 variables

Let be a thin plate or lamina with a variable density $\rho(x,y)$ where ρ is a continuous function on D a region of the xy-plane.
(the lamina occupies a region D).
the density ρ is defined as: $\rho(x,y) = \lim \dfrac{\Delta m}{\Delta A}$ when the limit is taken as the dimensions of the rectangle approach to 0.
Δm = mass of small rectangle that contains (x,y)
ΔA = area of ―――――――



- the total mass of the lamina is $\boxed{m = \iint_D \rho(x,y)\,dA}$

Remark: In electricity, if an electric charge is distributed over a region D and the charge of the density is given by $\sigma(x,y)$, $(x,y)\in D$.
then the total charge Q is $\iint_D \sigma(x,y)\,dA$

1) Center of mass: Find the coordinates of the center of mass of a lamina with density $\rho(x,y)$ that occupies a region D

→ we introduce first:

① the moment of the entire lamina about the x-axis, denoted M_x

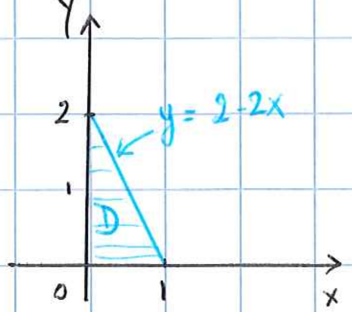
$$M_x = \iint_D y\,\rho(x,y)\,dA$$

② the moment of the entire lamina about the y-axis, denoted M_y

$$M_y = \iint_D x\,\rho(x,y)\,dA$$

the center of mass, denoted $(\bar{x},\bar{y})$ is $\boxed{\left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{\iint_D x\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}, \dfrac{\iint_D y\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}\right)}$

Example: Find the mass and the centr of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, $(0,2)$ with density $\rho(x,y) = 1+3x+y$.



First, describe the region D.

$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2-2x\}$ (type I domain)

$\quad = \{(x,y) \mid 0 \le y \le 2,\ 0 \le x \le \frac{2-y}{2}\}$ (type II domain)

- the total mass is $\iint_D \rho(x,y)\,dA = \int_0^1 \int_0^{2-2x} 1+3x+y \,dy\,dx$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_0^{2-2x} dx$$

$$= \int_0^1 -4x^2 + 4 \,dx = \left[-\frac{4x^3}{3} + 4x \right]_0^1 = \frac{8}{3}$$

- $\bar{x} = \frac{1}{m} \iint_D x\,\rho(x,y)\,dA$

$$= \frac{3}{8} \int_0^1 \int_0^{2-2x} x(1+3x+y)\,dy\,dx = \frac{3}{8} \int_0^1 \int_0^{2-2x} x + 3x^2 + xy \,dy\,dx$$

$$= \frac{3}{8} \int_0^1 \left[xy + 3x^2 y + x\frac{y^2}{2} \right]_0^{2-2x} dx = \frac{3}{8}$$

- $\bar{y} = \frac{3}{8} \int_0^1 \int_0^{2-2x} y\,\rho(x,y)\,dA = \frac{11}{16}.$

2) Moment of Inertia (or 2nd moment).

let D be a lamina with density $\rho(x,y)$.

the moment of inertia of the lamina about the x-axis is $\iint_D y^2 \rho(x,y)\,dA = I_x$

the moment of inertia of the lamina about the y-axis is $\iint_D x^2 \rho(x,y)\,dA = I_y$

the polar moment of inertia (about the origin) is denoted I_0

$$\boxed{I_0 = \iint_D (x^2+y^2)\,\rho(x,y)\,dA}$$