Chapter 14 : Vector Calculus.

14.1 Vector fields:

Definition: Let D be a set in $\mathbb{R}^2$ (resp. in $\mathbb{R}^3$).
A vector field on D is a function (vector function) that assigns to each point (x,y) (resp. (x,y,z)) in D a vector $F(x,y) \in \mathbb{R}^2$ (resp. $F(x,y,z) \in \mathbb{R}^3$)

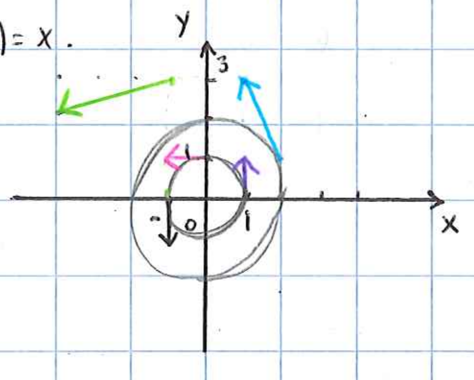
examples: ① Let f be a function of 2 or 3 variables.
$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ is a scalar field (kind of vector field)
ex if $f(x,y,z) = x^2y - y^3 + z^2$. then $\nabla f(x,y,z) = \langle 2xy, x^2-3y^2, 2z \rangle$

② Describe F by sketching some vectors $F(x,y)$ / $F(x,y,z)$ for:

1) $F(x,y) = -y\vec{i} + x\vec{j} = \langle -y, x \rangle$.

$F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ where P, Q are the components of F and functions of 2 variables $P(x,y) = -y$ and $Q(x,y) = x$.

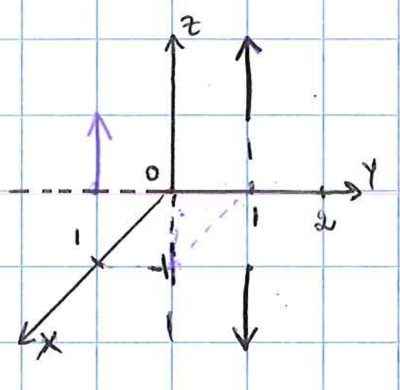
If $(x,y) = (1,0)$ then $F(1,0) = \langle 0,1 \rangle$
If $(x,y) = (2,1)$ then $F(2,1) = \langle -1,2 \rangle$
If $(x,y) = (-1,3)$ then $F(-1,3) = \langle -3,-1 \rangle$
$F(0,1) = \langle -1,0 \rangle$, $F(-1,0) = \langle 0,-1 \rangle$



It appears that each vector field is tangent to a circle with center the origin
How to prove that? Remark that $\langle x,y \rangle \cdot F(x,y) = \langle x,y \rangle \cdot \langle -y,x \rangle = 0$
It means that $F(x,y)$ is orthogonal to the vector position $\langle x,y \rangle$ and $|F(x,y)| = \sqrt{x^2+y^2}$

2) Describe $F(x,y,z) = z\vec{k} = \langle 0,0,z \rangle$.

$F(1,0,0) = \langle 0,0,0 \rangle$
$F(1,0,1) = \langle 0,0,1 \rangle$



Proposition A vector field is continuous on a region D if and only if each component is continuous on D.

__Definition__ : A vector field F is called a conservative vector field if it's the gradient of some function. (scalar function (with values in $\mathbb{R}$)).

In other words, if there exits a function f such that $\boxed{F = \nabla f}$.

In this case, f is called the potential function of

__example__ : the Gravitational force between 2 objects such that one of them is at the origin of mass M, the other one at the distance r of mass m exerted on this second object, acts toward the origin in the direction $-\dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$ where (x, y, z) is the position of the 2nd object

this force doted F is $F = \dfrac{mMG}{r^2} \cdot -\dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$ with $r = \sqrt{x^2+y^2+z^2}$

with G the gravitational constant.

F is a conservative force (means the energy is conserved during the travel)

ie there exits a function $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$ such that $F(x,y,z) = \nabla f(x,y,z)$

We will see in the next sections how to find f the potential function.

(how precisely solve
$$\begin{cases} f_x(x,y,z) = \dfrac{-mMG \cdot x}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} \\[2ex] f_y(x,y,z) = \dfrac{-mMG\, y}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}. \\[2ex] f_z(x,y,z) = \dfrac{-mMG\, z}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} \end{cases}$$