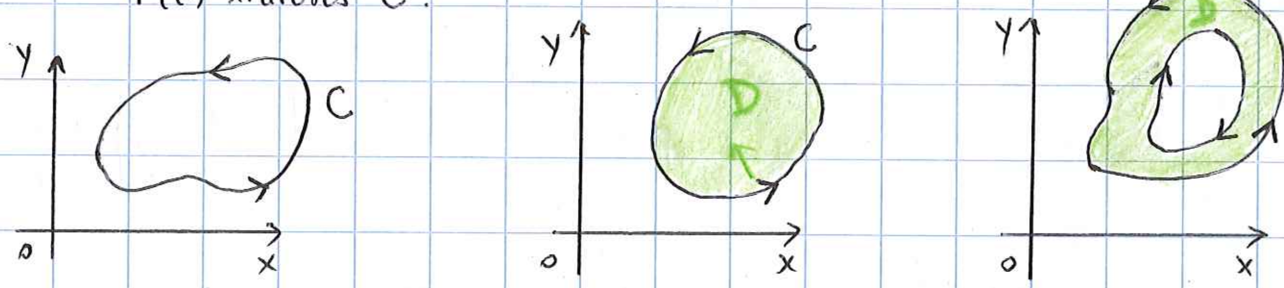
14.4 Green's theorem (in the plane)

This theorem gives a relationship between line integrals of a vect. field around simple closed curve and a double integral.

Convention: the positive orientation of a curve C refers to a single counterclockwise traversal of C.

examples: If C is given by $r(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$ then the region D bounded by C is always on the left as the point $r(t)$ traverses C.



Notation: we denote ∂D the boundary of D, equivalent to C a closed curve

Theorem (Green's theorem): Let C be a positively oriented, piecewise smooth simple closed curve in the plane & D be the region bounded by C.
If P and Q have continuous first partial derivatives on an open region that contains D then

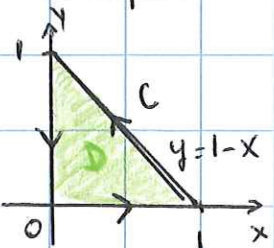
$$\boxed{\int_{C=\partial D} P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(Q_x(x,y) - P_y(x,y)\right) dA}$$

$$\| $$

notation to say positively oriented $\longrightarrow$ $\oint_C F \cdot dr$ with $F = \langle P, Q \rangle$.

Remark: If C is negatively oriented $\int_C F \cdot dr = -\oint_C F \cdot dr$.

examples: ① Evaluate $\int_C x^4\,dx + xy\,dy$ where C is the triangular curve



C bounds D:

if we want to compute directly $\int_C x^2\,dx + xy\,dy$, we need to parametrize 3 segment lines: it's easier by using Green's theorem since C is a closed and simple curve and $P(x,y) = x^4$ & $Q(x,y) = xy$ are polynomial functions.

Green's theorem says: $\int_C x^4\,dx + xy\,dy = \iint_D Q_x - P_y\,dA = \iint_D y - 0\,dA$

where D (seen as a type I domain) $= \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1-x\}$

$$\int_C x^4\,dx + xy\,dy = \int_0^1\int_0^{1-x} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2}\,dx = \left[-\frac{(1-x)^3}{6}\right]_0^1 = \frac{1}{6}$$

② Applications of Green's th: computing areas

Theorem: Let D be a region in the plane bounded by C $(= \partial D)$ a curve (simple & closed)

by Green's theorem: $\underset{\text{area of } D}{\underset{\parallel}{A(D)}} = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx.$

idea (of the proof): $A(D) = \iint_D 1\,dA = \iint_D Q_x - P_y\,dA$ so we identify $Q_x - P_y = 1$

we have $Q_x = 1$ and $P_y = 0$ or $Q_x = 0$ and $P_y = -1$ or $Q_x = \frac{1}{2}$ & $P_y = -\frac{1}{2}$

↓ integration

$Q = x$ & $P = \underset{0}{\underset{\parallel}{cst}}$ or $Q = \underset{0}{\underset{\parallel}{cst}}$ & $P = -y$ or $Q = \frac{x}{2}$ & $P = -\frac{y}{2}$.

example: Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

the area is $A(D) = \iint_D dA = \frac{1}{2}\int_C x\,dy - y\,dx$. where C is the ellipse parametrized by $r(t) = \langle 2\cos t, 3\sin t\rangle = \langle x(t), y(t)\rangle$

for $0 \leq t \leq 2\pi$ so $A(D) = \frac{1}{2}\int_0^{2\pi}[2\cos t.(3\cos t) - 3\sin t(-2\sin t)]\,dt = \frac{1}{2}\int_0^{2\pi} 6\,dt = 6\pi$