14.9 The Divergence theorem.

Theorem (Gauss th. or Divergence th.): Let E be a simple solid regions whose the boundary surface S has positive orientation (⇔ outward).
Let F be a continuous vector field on an open region that contains E.
Then

$$\boxed{\oiint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} F \, dV}$$

Interpretation: the divergence is the tendancy of the vect. field to diverge. (to move toward) from the point.

If we see F the vector field as a velocity fluid or gaz:
- If $\operatorname{div} \vec{F} > 0$ then the divergence measures the expansion of the fluid or gas.
- If $\operatorname{div} \vec{F} < 0$ then the fluid or gas contracts

↳ Why? From the previous theorem, suppose that $\operatorname{div} F = cst = K$
then $\oiint_S \vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div} F}_{K} \, dV = \underbrace{\operatorname{div} F}_{K} \iiint_E dV$ so. $\operatorname{div} F = \dfrac{\text{flux of } F}{\text{Volume}}$

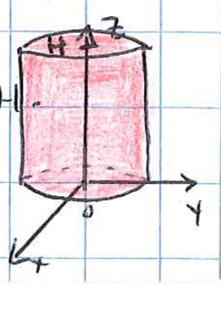
Gauss / Divergence theorem says that
- the divergence is the outgoing flux per volume

or
- the total expansion of the fluid inside some 3-dimensional region E equals to the total flux of the fluid out of the boundary S.

⇒ theorem is useful to convert a surface integral into a volume integral!

examples: ① Let $E = \{(x,y,z) \mid x^2 + y^2 \leq R^2, 0 \leq z \leq H\}$.
Find the flux of $F(x,y,z) = \langle 1+x, 3+y, z-10 \rangle$ over $S = \partial E$.

E is the portion of the cylinder $x^2+y^2 = R^2$ between $z=0$ & $z=H$.
S is a closed surface so oriented outward.



By using Gauss theorem, $\oiint_S F.d\vec{S} = \iiint_E \operatorname{div} F\, dV$. we just need to compute div F

$\operatorname{div} F = \frac{\partial}{\partial x}(1+x) + \frac{\partial}{\partial y}(3+y) + \frac{\partial}{\partial z}(z-10) = 1+1+1 = 3.$

So. $\oiint_S \vec{F}.d\vec{S} = 3\iiint_E dV = 3\int_0^{2\pi}\int_0^R\int_0^H r\,dz\,dr\,d\theta = 3.2\pi.H.\left[\frac{r^2}{2}\right]_0^R = 3\pi R^2 H$

Switch to cylindrical coordinate $E \rightsquigarrow \tilde{E} = \{(r,\theta,z) \mid 0\le r\le R, 0\le\theta\le 2\pi, 0\le z\le H\}$

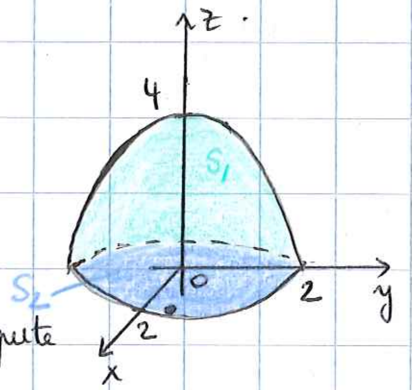
Remark: If $F(x,y,z) = \langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3}\rangle$ then div $F = 1$

So $\oiint_S F.d\vec{S} = \iiint_E dV = \text{volume}(E)$.

② Let E be the solid bounded by the paraboloid $z = 4-x^2-y^2$ & the xy-plane.

Evaluate $I = \iint_S \underbrace{\langle x^3, 2xz^2, 3y^2z\rangle}_{F(x,y,z).}\cdot d\vec{S}$ if

a) S is the boundary of E:



Denote S_1 the paraboloid $z=4-x^2-y^2$ and S_2 the disc intersection of E and the xy-plane.

If we want to compute directly I, we need to compute 2 surface integrals. $\oiint_S \vec{F}.d\vec{S} = \iint_{S_1} F.d\vec{S} + \iint_{S_2} F.d\vec{S}.$

We use Gauss theorem: $I = \iiint_E \operatorname{div}(\langle x^3, 2xz^2, 3y^2z\rangle)\,dV = \iint_D\left(\int_0^{4-x^2-y^2} 3x^2+3y^2\,dz\right)dA$

where $E = \{(x,y,z) \mid x^2+y^2\le 4, 0\le z\le 4-x^2-y^2\}$ and $D = \{x^2+y^2\le 4\} = S_2$.

$I = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 3r^2.r\,dz\,dr\,d\theta = 2\pi\int_0^2 3r^3(4-r^2)\,dr = 32\pi.$

in cylindrical coord. $E \rightsquigarrow \tilde{E} = \{(r,\theta,z) \mid 0\le r\le 2, 0\le\theta\le 2\pi, 0\le z\le 4-r^2$

b) If S is the part of the paraboloid between $z=0$ & $z=4$. ie $\oiint_{S_1} F.d\vec{S}$

From a) we know. $\oiint_S F.d\vec{S} = \iint_{S_1} F.d\vec{S} + \iint_{S_2} F.d\vec{S} = 32\pi$.

So we just need to compute $\iint_{S_2} F.d\vec{S}$ (easy since $S_2 = \{(x,y) \mid x^2+y^2 \leq 4\}$)

a normal vector outward to S_2 is $\langle 0,0,-1\rangle$.

and $F_{|S_2}(x,y,z) = \langle x^3, 0, 0\rangle$ since $z=0$ on S_2.

so $\iint_{S_2} F.d\vec{S} = \iint_{S_2} \langle x^3,0,0\rangle \cdot \langle 0,0,-1\rangle \, dS = 0$.

then $\oiint_S F.d\vec{S} - \oiint_{S_2} F.d\vec{S} = \oiint_{S_1} F.d\vec{S} = 32\pi$.

(3) $I = \iint_S \text{curl } F.d\vec{S}$ if S is the boundary of $E = \{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1\}$ for some vector field F.

$I = \oiint_S \text{curl } F.d\vec{S} = \iiint_E \underbrace{\text{div}(\text{Curl } F)}_{= 0} \, dV$ by Gauss th.

$= 0$

Remark: Be careful: we cannot use Stokes theorem here: S has no boundary!

Summary:

object	param	boundary	param	formulas
Solid E	3.	closed surface S	2	Divergence th: $\iiint_E \text{div } F \, dV = \oiint_S F.d\vec{S}$.
Surface S	2	curve C	1	Stokes' th: $\oiint_S \text{Curl } F.d\vec{S} = \int_C \vec{F}.d\vec{r}$
plane region D.		Curve C		Green - Reimann: $\int_C F.dr = \iint_D (Q_x - P_y)\, dA$
Closed Surface S		no boundary		