ERRATUM TO: ON L^2-ESTIMATES FOR $\bar{\partial}$ ON A PSEUDOCONVEX DOMAIN IN A COMPLETE KÄHLER MANIFOLD WITH POSITIVE HOLOMORPHIC BISECTIONAL CURVATURE

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ABSTRACT. We correct some results in our earlier paper, about the Diederich-Fornaess exponent for the distance function in a complete Kähler manifold with positive holomorphic bisectional curvature. The main concern is about the definition of the function τ_Ω. This was corrected in the author's thesis. The main results on the existence of Diederich-Fornaess exponent and the L^2-estimates for $\bar{\partial}$ operator still hold.

We use the notations of [1].

Definition 4.2 [1] needs to be remplaced by the following:

Definition 4.2. *Let U be a neighborhood of $\partial\Omega$. Then*

$$\tau_\Omega(x) = \|\Lambda_x\|^2_{\mathcal{T}^*_x U, \widetilde{h_x}}, \quad \forall x \in U \cap \Omega.$$

Proposition 4.1 [1] needs to be removed. These corrections appear because of the non-continuity of the function τ_Ω on the boundary $\partial\Omega$.
We explain this confusion in the following paragraph now. This also implies different corrections in the results of section 5 in [1].

The tools used to prove Proposition 4.1 [1] were the Cauchy-Schwarz inequality combined with a property on the behavior of the Levi form close to the boundary but those cannot be used in this situation for the following reasons:

Let $z \in U \cap \overline{\Omega}$. Since the eigenvalue $i\partial\bar{\partial}_z(-\delta_{\partial\Omega})(L_n, \overline{L_n})$ may be negative ($\delta_{\partial\Omega}$ is not plurisubharmonic), the Cauchy-Schwarz inequality needs to be applied on $Q_z = \delta_{\partial\Omega}(z).i\partial\bar{\partial}_z(-\log\delta_{\partial\Omega})$ (since $-\log\delta_{\partial\Omega}$ is strictly plurisubharmonic on Ω) as follows

$$|i\partial\bar{\partial}_z(-\delta_{\partial\Omega})(L_n, \overline{L_i})| \leq Q_z(L_i, \overline{L_i})^{\frac{1}{2}} Q_z(L_n, \overline{L_n})^{\frac{1}{2}}, \tag{1}$$

since $Q_z(L_n, \overline{L_i}) = i\partial\bar{\partial}_x(-\delta_{\partial\Omega})(L_n, \overline{L_i})$.

Unfortunately, this inequality is not sufficient to obtain directly the fact that $\tau_\Omega < 1$ on $U \cap \overline{\Omega}$.

1991 *Mathematics Subject Classification.* 32W05-32T35-32Q10.
Key words and phrases. Pseudoconvex domains, Plurisubharmonic exhaustion functions, Reach, Diederich-Fornaess exponent.

This may be explained by the positivity of the curvature:

If the curvature of the manifold is zero (for example in $\mathbb{C}^n$), then τ_Ω is equal to 0 on the boundary from (1) since for $p \in \mathcal{W}$ a weakly pseudoconvex boundary point and $w \in \mathcal{N}_p$, we have

$$\frac{d}{d\varepsilon}\langle i\partial\bar{\partial}(-\delta_{\partial\Omega}(z_\varepsilon)), w \wedge \bar{w}\rangle = 0,$$

where $z_\varepsilon = p + \varepsilon \nabla \delta_{\partial\Omega}(p)$ is a point ε-equidistant to the boundary.

This implies $i\partial\bar{\partial}_p(-\delta_{\partial\Omega})(L_n, \overline{L_i}) = 0$ for every $L_i \in Ker(\mathcal{L}_p\delta_{\partial\Omega})$ (see [4] or the *Riccati equation* (*Curvature Equations*) in [II, 4.2, [3]]).

This argument doesn't hold anymore when the curvature is positive. We cannot prove the continuity of τ_Ω on $\partial\Omega$ by extending it by 0 on the boundary.

Hence, Corollary 4.1 in [1], converted into a Proposition is corrected as follows :

Proposition 4.1. *Let (X, ω) be a complete Kähler manifold with positive holomorphic bisectional curvature and $\Omega \Subset X$ be a pseudoconvex domain with $\mathcal{C}^2$ boundary. Then there exists a neighborhood U of $\partial\Omega$ and a positive constant c such that*

$$\forall z \in U \cap \Omega, \quad \tau_\Omega(z) \leq c < 1.$$

Proof.

Since $\partial\Omega$ is $\mathcal{C}^2$, reach$(\Omega^c) \geq a > 0$ for some $a > 0$. So there exists a neighborhood U of $\partial\Omega$ on which $\delta_{\partial\Omega}$ is also of class $\mathcal{C}^2$ and on which the function τ_Ω is well defined. By contradiction, suppose there exists a sequence of points $z_k \in \Omega$ convergent to $p \in \partial\Omega$ such that $\lim_{k\to+\infty} \tau_\Omega(z_k) = 1$. Then, thanks to Proposition 7.2 [1], there doesn't exist an exponent $\eta > 0$ such that $-\delta^\eta_{\partial\Omega}$ is strictly plurisubharmonic on Ω, which is a contradiction with [2].

□

We also correct Theorem 5.1 [1]:

Theorem 5.1. *Let Ω be a relatively compact pseudoconvex domain with $\mathcal{C}^2$ boundary in a complete Kähler manifold with positive holomorphic bisectional curvature. Let $\varepsilon > 0$ and V_ε the ε-equidistant neighborhood of the boundary $\partial\Omega$. Then*

$$t(\partial\Omega) \geq 1 - \inf_{\varepsilon} \sup_{V_\varepsilon \cap \Omega} \tau_\Omega^{\frac{1}{2}}. \tag{2}$$

Proof of Theorem 5.1.

We prove this Theorem into two steps. First, we use Ohsawa and Sibony's idea [2] and precise a first estimation, thanks to Proposition 5.1 [1]. Secondly, we deduce an estimation of the order of plurisubharmonicity.

Step 1: Prove that on a small enough neighborhood U of Ω, for all $0 < \alpha < \inf_{U\cap\Omega}\gamma.(1-\tau_\Omega^{\frac{1}{2}})$, the function $-\delta_{\partial\Omega}^\alpha$ is strictly plurisubharmonic on $U\cap\Omega$.

Remark that we have $\inf_{U\cap\Omega}\gamma.(1-\tau_\Omega^{\frac{1}{2}}) > 0$:
By Proposition 4.1, there exists a neighborhood U' of $\partial\Omega$ and a constant $c' > 0$ such that $\tau_\Omega(x) \leq c' < 1$ for $x \in U'\cap\Omega$. Hence, $c_0(x) := 1-\tau_\Omega^{\frac{1}{2}} \geq 1-\sqrt{c'} := c > 0$ on $U'\cap\Omega$.

Moreover, the fonction γ being continuous on $U'\cap\bar{\Omega}$ and equal to 1 on $\partial\Omega$, reducing U' if need be, $\inf_{U'\cap\Omega}\gamma > 0$.

Let $(U_i)_{1\leq i\leq N}$ be a small enough covering of $\partial\Omega$. By Proposition 5.1 [1], for all $x \in U_i\cap\Omega$,

$$Q_x(v) \geq c_0(x).\left(\sum_{i=1}^{n-1} C_\Omega \delta_{\partial\Omega}^2(x)|v_i|^2 + \gamma(x).|v_n|^2\right). \tag{3}$$

By computing $i\partial\bar{\partial}(-\delta_{\partial\Omega}^\alpha)$ for $\alpha > 0$ and by Proposition 5.1 [1], we deduce a sufficient condition on α in order for the function $-\delta_{\partial\Omega}^\alpha$ to be strictly plurisubharmonic:

From (3), we have on $U\cap\Omega$ for $U = \left(\bigcup_{1\leq i\leq N} U_i\right)\cap U'$:

$$\frac{1}{\delta_{\partial\Omega}^2(x)}Q_x(v) + C_\Omega|v_n|^2 \geq c_0(x)\left(\sum_{j=1}^{n-1} C_\Omega|v_j|^2 + \left(\frac{C_\Omega}{c_0(x)} + \frac{\gamma(x)}{\delta_{\partial\Omega}^2(x)}\right)|v_n|^2\right) \tag{4}$$

$$\Longleftrightarrow \frac{1}{\delta_{\partial\Omega}(x)}\sum_{i,j=1}^{n} a_{ij}v_i\bar{v_j} + \frac{1}{\delta_{\partial\Omega}^2}\|\partial\delta_{\partial\Omega}\|^2|v_n|^2 \geq c_0(x)\left(\sum_{j=1}^{n-1} C_\Omega|v_j|^2 + \left(\frac{C_\Omega}{c_0(x)} + \frac{\gamma(x)}{\delta_{\partial\Omega}^2(x)}\right)|v_n|^2\right).$$

Let $\alpha > 0$ and $v \in T^{(1,0)}U$,

$$\begin{aligned}
\langle i\partial\bar{\partial}(-\delta_{\partial\Omega}^\alpha), v\wedge\bar{v}\rangle &= \left(\langle \alpha\delta_{\partial\Omega}^{\alpha-1} i\partial\bar{\partial}(-\delta_{\partial\Omega}) - \alpha(\alpha-1)\delta_{\partial\Omega}^{\alpha-2} i\partial\delta_{\partial\Omega}\wedge\bar{\partial}\delta_{\partial\Omega}, v\wedge\bar{v}\rangle\right) \\
&= \alpha\delta_{\partial\Omega}^\alpha\left(\left\langle \frac{(1-\alpha)}{\delta_{\partial\Omega}^2} i\partial\delta_{\partial\Omega}\wedge\bar{\partial}\delta_{\partial\Omega}, v\wedge\bar{v}\right\rangle + \left\langle \frac{i\partial\bar{\partial}(-\delta_{\partial\Omega})}{\delta_{\partial\Omega}}, v\wedge\bar{v}\right\rangle\right) \\
&= \alpha\delta_{\partial\Omega}^\alpha\left(\frac{1}{\delta_{\partial\Omega}}\sum_{j,i=1}^{n} a_{ji}v_j\bar{v}_i + \frac{(1-\alpha)}{\delta_{\partial\Omega}^2}\|\partial\delta_{\partial\Omega}\|^2|v_n|^2\right) \\
&= \alpha\delta_{\partial\Omega}^\alpha\left(\frac{1}{\delta_{\partial\Omega}^2}Q_x(v) + C_\Omega|v_n|^2 - \frac{\alpha}{\delta_{\partial\Omega}^2}\|\partial\delta_{\partial\Omega}\|^2|v_n|^2\right) \\
&\geq c_0(x)\alpha\delta_{\partial\Omega}^\alpha\left(\sum_{j=1}^{n-1} C_\Omega|v_j|^2 + \left(\frac{C_\Omega}{c_0(x)} + \frac{\gamma(x)}{\delta_{\partial\Omega}^2} - \frac{\alpha.\|\partial\delta_{\partial\Omega}\|^2}{c_0(x).\delta_{\partial\Omega}^2}\right)|v_n|^2\right) \text{ by (4)} \\
&\geq C_\alpha(x).\delta_{\partial\Omega}^\alpha\sum_{i=1}^{n}|v_j|^2, \quad \text{si } \frac{C_\Omega}{c_0(x)} + \frac{\gamma(x)}{\delta_{\partial\Omega}^2} - \frac{\alpha.\|\partial\delta_{\partial\Omega}\|^2}{c_0(x).\delta_{\partial\Omega}^2} > 0, \qquad (5)
\end{aligned}$$

where C_α is a positive constant dependent on α.

Assumption (5) is equivalent to $0 < \alpha < \dfrac{\gamma(x).c_0(x) + C_\Omega.\delta^2_{\partial\Omega}(x)}{\|\partial\delta_{\partial\Omega}\|^2}$.

As $\|\partial\delta_{\partial\Omega}(x)\| = 1$ on U small enough, if

$$0 < \alpha < \inf_{x\in U\cap\Omega}(\gamma(x).c_0(x) + C_\Omega.\delta^2_{\partial\Omega}(x)) = \inf_{x\in U\cap\Omega}(\gamma(x).c_0(x))$$

then

$$\langle i\partial\bar\partial(-\delta^\alpha_{\partial\Omega}), v\wedge\bar v\rangle \geq C_\alpha\delta^\alpha_{\partial\Omega}\|v\|^2 \qquad \text{on } U\cap\Omega. \tag{6}$$

Step 2: Estimate of the order of plurisubharmonicity $t(\partial\Omega)$.

Denote V_ε the ε-equidistant neighborhood of the boundary $\partial\Omega$ and U the previous neighborhood. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $V_\varepsilon \subset U$.

By Step 1, for all $0 < \alpha < \inf_{x\in V_\varepsilon\cap\Omega}\gamma(x).c_0(x)$, the function $-\delta^\alpha_{\partial\Omega}$ is strictly plurisubharmonic on $V_\varepsilon\cap\Omega$.

So,

$$\forall\, 0 < \varepsilon < \varepsilon_0, \quad t(\partial\Omega) \geq \left(\inf_{x\in V_\varepsilon\cap\Omega}\gamma(x).c_0(x)\right).$$

And,

$$t(\partial\Omega) \geq \sup_{0<\varepsilon<\varepsilon_0}\left(\inf_{x\in V_\varepsilon\cap\Omega}\gamma(x).c_0(x)\right).$$

But,

$$\sup_\varepsilon\left(\inf_{x\in V_\varepsilon\cap\Omega}\gamma(x).c_0(x)\right) = (\sup_\varepsilon\inf_{x\in V_\varepsilon\cap\Omega}\gamma(x)).(1 - \inf_\varepsilon\sup_{x\in V_\varepsilon\cap\Omega}\tau_\Omega^{\frac{1}{2}}(x)).$$

Recall that γ is continuous $U\cap\bar\Omega$, so $\sup_\varepsilon\inf_{x\in V_\varepsilon\cap\Omega}\gamma(x) = \inf_{x\in\partial\Omega}\gamma(x) = 1$.

Hence,

$$t(\partial\Omega) \geq 1 - \inf_\varepsilon\sup_{x\in V_\varepsilon\cap\Omega}\tau_\Omega^{\frac{1}{2}}(x).$$

□

The correction of Corollary 4.1 in [1] implies the following modification of Corollary 5.1 in [1] :

Corollary 5.1. *Let (X,ω) be a complete Kähler manifold with positive holomorphic bisectional curvature and $\Omega \Subset X$, a pseudoconvex domain. Let $(\Omega_k)_{k\geq1}$ be an exhaustion by smooth pseudoconvex domains of Ω such that*

$$\sup_k \tau_{\Omega_k} \leq c < 1,$$

for a positive constant c independent of k. Then there exists $\alpha > 0$ such that for all $k \geq 1$, there exists a neighborhood U_k of $\partial\Omega_k$ such that $-\delta^\alpha_{\Omega_k} \in s - PSH(U_k\cap\Omega_k)$.

Proof. Since for all $k \geq 1$, $\Omega_k \Subset \Omega$ is a pseudoconvex domain with $\mathcal{C}^2$ boundary. We also have

$$i\partial\bar{\partial}(-\log \delta_{\partial\Omega_k}) \geq C_\Omega \omega, \quad \text{on } \Omega_k.$$

By using the proof of Proposition 5.1 on Ω_k, there exists a neighborhood U_k on which any α_k with

$$0 < \alpha_k \leq (1 - \tau_{\Omega_k}^{\frac{1}{2}}) \min(C_\Omega, \gamma_k),$$

is a Diederich-Fornaess exponent for the distance function δ_{Ω_k} on $U_k \cap \Omega_k$.

Since γ_k is continuous and equal to 1 on $\partial\Omega_k$, reducing U_k if need be, there exists a positive constant b such that for every $k \geq 1$, $\gamma_k \geq 1 - b$ on $U_k \cap \Omega_k$. Moreover, by assumption, there exists a positive constant $\tilde{c}$ such that for every $k \geq 1$, $1 - \tau_{\Omega_k}^{\frac{1}{2}} \geq \tilde{c} > 0$ on $U_k \cap \Omega_k$. So, there exists an exponent α such that for every $k \geq 1$, there exists a neighborhood U_k of $\partial\Omega_k$ such that $-\delta_{\partial\Omega_k}^{\alpha} \in s - PSH(U_k \cap \Omega_k)$.

□

Aknowledgment: The author would like to thank J-P. Demailly and M. Adachi for pointing out some errors in the paper [1].

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