Foundations of statistical theory

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Outline

1. Course material in relation to the real world
   - Examples of statistical applications
   - The course
2. Probability theory
   - Distributions and random variables
   - Transformations
   - Generating and characteristic functions
   - Convergence of sequences of random variable
3. Statistics
   - Point estimation
   - Testing
   - Interval estimation

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Examples??

The course is theoretical but we will regularly give examples of applications. . .
One can study statistical theory in isolation - as a mathematical topic, but most commonly statistics relates to real life, making it harder!

Simple examples

From the Statistical Computing Centre (TMH)

- Handball
- Stutter
- Breathing/chest width
-

Linear models, random effects, test assumptions, interpret results. . .
Essential: Know R

More examples

- harvest control rules for fish stocks
- financial derivatives
- dendrograms in ecology
- ptarmigan nonlinear models

Bayesian statistics, mix of statistics and OR, descriptive statistics, bootstrap, nonlinear models, stochastic differential equations . . .
Essential: Proficiency in programming
The content of this course is used in all of the above applications!

Complex examples

- fMRI
- gene expression
- meteorology
- multispecies fisheries models

Complex examples require much more theoretical knowledge...
Start to require parallel computing...
Need to consider simultaneous inference
Model verification becomes very important...

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This course

- Forms theoretical basis
- Gives estimation methodology
- How to derive confidence statements
- Optimal tests of hypotheses
- General methods for applying large-sample inference
- Some applications

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Probability distributions

- Discrete
- Continuous
- Multivariate

Possibly include some measure-theoretical aspects
Specific distributions: normal, gamma, ...

Random variables

- As functions, ...
- Expected value, variance, ...
- Independence, covariance, correlation, ...
- Random sample: iid

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Examples

- $X, Y \sim Gamma(\alpha, \beta),\ \text{indep} \Rightarrow X + Y \sim ?$
- $X \sim n(0, 1) \Rightarrow X^2 \sim ?$
- $\mathbf{X} \sim n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (vector r.v.), A a matrix $\Rightarrow AX \sim ?$

General multivariate versions, Jacobian, ...
Specific distribution: MVN

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The moment generating function

$$M(t) = E\left[e^{tX}\right]$$

The characteristic function

$$M(t) = E\left[e^{itX}\right]$$

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Limit Theorems

$X_1, X_2, \ldots$ iid

- Central Limit Theorem
- Slutsky . . .

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General criteria for estimators

- Likelihood function
- Bias/Variance
- MSE
- MINVUE/UMVUE/BLUE

Properties of distributions/statistics/estimators

- Completeness
- Sufficience

Methods for deriving estimators

- Likelihood function
- Method of moments
- ...

Note: Will get bounds on how good it can get!

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Hypothesis test

Commonly want to test hypotheses on unknown parameters...
Reject if data are not in accordance with hypothesis...
Need assumptions on data...
Guarantee low probability of incorrect rejection (Type I error)...
In applications, can only reject, not accept...

Methods for deriving tests

- Likelihood ratio test
- ...

Note: Can get optimal tests!

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Methodology

- inverting tests...
- pivoting...

Evaluate the quality of these intervals...

Summary

- Probability theory (for techniques)
- Convergence
- Estimation
- Testing
- Confidence intervals
- Other topics
- Next
  - Optionally add some later sections
  - Bootstrapping/jackknife/permutation tests
- Result:
  - Solid basis for statistical theory
  - Foundation for other courses in stats
  - Foundation for further studies
  - This is a graduate level course