

Háskóli Íslands	09.12.21	Raunvísindadeild
Laugardagur	Grundvöllur tölfræðinnar 15. janúar 2010	kl 09:00-12:00
Leyfileg hjálpargögn: Skrif- færi og reiknivél. Jöfnublöðum er dreift með prófinu.		Dæmin vega jafnt.

Í dæmum 1-4 verður litið á úrtak (óháðar, einsdreifðar hendingar), X_1, \dots, X_n sem lúta hliðraðri veldisvísisdreifingu,

$$f_{\theta}(x) = \begin{cases} e^{-(x-\theta)} & \text{ef } x > \theta \\ 0 & \text{annars} \end{cases}$$

og við látum $X_{(1)} := \min\{X_1, \dots, X_n\}$.

1. (a) Finnið þéttifall, væntigildi og dreifni $X_{(1)}$.

(b) Skrifum $S_n = X_{(1)}$ til að auðkenna breytilega sýnastærð. Veljið talnarunur a_n og b_n þannig að

$$\frac{S_n - a_n}{b_n}$$

hafi markgildi í dreifingu (og markgildisdreifingin hafi stöð sem er ekki bara í einni tölu).

2. Notum forsendurnar að ofan

- (a) Finnið nægjanlega reiknihendingu, T .
- (b) Finnið MLE fyrir θ .
- (c) Er T fullkomin?
- (d) Finnið besta óbjagaða metilinn (UMVUE) fyrir θ .

[Athugið að hér getur verið til einföldunar að byrja á að skoða dreifingu $X_i - \theta$ áður er t.d. er reynt að reikna væntigildi og álíka.]

3. Sömu forsendur og áður.

- (a) Er MLE fyrir θ óbjagaður?
- (b) Er MLE fyrir θ mótsagnalaus?
- (c) Er MLE fyrir θ^2 óbjagaður?
- (d) Er MLE fyrir θ^2 mótsagnalaus?
- (e) Finnið MINVUE metil fyrir θ^2 .

[Athugið: Hér þarf að rökstyðja svörin en **ekki þarf að reikna** heldur er hægt að svara hverjum lið með einni setningu (t.d. tilvísun í þekkta niðurstöðu) eða jöfnu]

4. Lítið aftur á hendingarnar og þéttiföllin að ofan.

(a) Finnið sennileikaprófið (LRT) fyrir $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$.

(b) Finnið sterkasta prófið (UMP) fyrir $H_0 : \theta = \theta_0$ gegn $H_1 : \theta = \theta_1$. Gerið ráð fyrir að $\theta_1 > \theta_0 > 0$. Finnið sem einfaldasta framsetningu á prófinu og lýsið, hvernig finna má þann fasta sem með þarf til að prófið verði af tilteknu stigi, ef hægt er.

Aukadæmi: Látum hendingar X_1, \dots, X_n vera úrtak (óháðar, einsdreifðar) sem lúta dreififalli F . Gerum ráð fyrir að F sé samfellt og áhugi sé á að fá öryggismörk fyrir stika, ξ , sem lýsir miðgildi dreifingarinnar, þ.e. $F(\xi) = 1/2$, en að öðru leyti sé F óþekkt.

Finnið heiltölur i og j þannig að bilið $(X_{(i)}, X_{(j)})$, byggt á raðhendingum, myndi (a.m.k.) $100(1 - \alpha)\%$ öryggismörk fyrir ξ .

[Byrja má á að finna t.d. neðra mark. Þá er gott að byrja á að gera sér grein fyrir að ef T er hending sem lýsir fjölda X_k neðan ξ ,

$$T = \sum_k 1_{\{X_k \leq \xi\}},$$

þá er atburðurinn $T \geq l$ sá sami og að $X_{(l+1)} \leq \xi$ m.ö.o.

$$\{T \geq l\} = \{X_{(l+1)} \leq \xi\},$$

en T hefur augljósa dreifingu. Athugið að út kemur lýsing á, hvernig finna má i og j en lausnirnar eru ekki á lokuðu formi.]

Table of Common Distributions

Discrete Distributions

Bernoulli(p)

pmf $P(X = x|p) = p^x(1-p)^{1-x}; \quad x = 0, 1; \quad 0 \leq p \leq 1$

mean and variance $EX = p, \quad \text{Var } X = p(1-p)$

mgf $M_X(t) = (1-p) + pe^t$

Binomial(n, p)

pmf $P(X = x|n, p) = \binom{n}{x} p^x (1-p)^{n-x}; \quad x = 0, 1, 2, \dots, n; \quad 0 \leq p \leq 1$

mean and variance $EX = np, \quad \text{Var } X = np(1-p)$

mgf $M_X(t) = [pe^t + (1-p)]^n$

notes Related to Binomial Theorem (Theorem 3.2.2). The *multinomial* distribution (Definition 4.6.2) is a multivariate version of the binomial distribution.

Discrete uniform

pmf $P(X = x|N) = \frac{1}{N}; \quad x = 1, 2, \dots, N; \quad N = 1, 2, \dots$

mean and variance $EX = \frac{N+1}{2}, \quad \text{Var } X = \frac{(N+1)(N-1)}{12}$

mgf $M_X(t) = \frac{1}{N} \sum_{i=1}^N e^{it}$

Geometric(p)

pmf $P(X = x|p) = p(1-p)^{x-1}; \quad x = 1, 2, \dots; \quad 0 \leq p \leq 1$

mean and variance $EX = \frac{1}{p}, \quad \text{Var } X = \frac{1-p}{p^2}$

<i>mgf</i>	$M_X(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$
<i>notes</i>	$Y = X - 1$ is negative binomial(1, p). The distribution is <i>memoryless</i> : $P(X > s X > t) = P(X > s - t)$.

Hypergeometric

<i>pmf</i>	$P(X = x N, M, K) = \frac{\binom{M}{x} \binom{N-M}{K-x}}{\binom{N}{K}}; \quad x = 0, 1, 2, \dots, K;$ $M - (N - K) \leq x \leq M; \quad N, M, K \geq 0$
<i>mean and variance</i>	$EX = \frac{KM}{N}, \quad \text{Var } X = \frac{KM}{N} \frac{(N-M)(N-K)}{N(N-1)}$
<i>notes</i>	If $K \ll M$ and N , the range $x = 0, 1, 2, \dots, K$ will be appropriate.

Negative binomial(r, p)

<i>pmf</i>	$P(X = x r, p) = \binom{r+x-1}{x} p^r (1-p)^x; \quad x = 0, 1, \dots; \quad 0 \leq p \leq 1$
<i>mean and variance</i>	$EX = \frac{r(1-p)}{p}, \quad \text{Var } X = \frac{r(1-p)}{p^2}$
<i>mgf</i>	$M_X(t) = \left(\frac{p}{1-(1-p)e^t} \right)^r, \quad t < -\log(1-p)$
<i>notes</i>	An alternate form of the pmf is given by $P(Y = y r, p) = \binom{y-1}{r-1} p^r (1-p)^{y-r}, y = r, r+1, \dots$. The random variable $Y = X + r$. The negative binomial can be derived as a gamma mixture of Poissons. (See Exercise 4.34.)

Poisson(λ)

<i>pmf</i>	$P(X = x \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}; \quad x = 0, 1, \dots; \quad 0 \leq \lambda < \infty$
<i>mean and variance</i>	$EX = \lambda, \quad \text{Var } X = \lambda$
<i>mgf</i>	$M_X(t) = e^{\lambda(e^t-1)}$

Continuous Distributions

Beta(α, β)

pdf $f(x|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 \leq x \leq 1, \quad \alpha > 0, \quad \beta > 0$

mean and variance $EX = \frac{\alpha}{\alpha+\beta}, \quad \text{Var } X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

mgf $M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}$

notes The constant in the beta pdf can be defined in terms of gamma functions, $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Equation (3.2.18) gives a general expression for the moments.

Cauchy(θ, σ)

pdf $f(x|\theta, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1 + \left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty; \quad -\infty < \theta < \infty, \quad \sigma > 0$

mean and variance do not exist

mgf does not exist

notes Special case of Student's t , when degrees of freedom = 1. Also, if X and Y are independent $n(0, 1)$, X/Y is Cauchy.

Chi squared(p)

pdf $f(x|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2}; \quad 0 \leq x < \infty; \quad p = 1, 2, \dots$

mean and variance $EX = p, \quad \text{Var } X = 2p$

mgf $M_X(t) = \left(\frac{1}{1-2t} \right)^{p/2}, \quad t < \frac{1}{2}$

notes Special case of the gamma distribution.

Double exponential(μ, σ)

pdf $f(x|\mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$

mean and variance $EX = \mu, \quad \text{Var } X = 2\sigma^2$

mgf $M_X(t) = \frac{e^{\mu t}}{1-(\sigma t)^2}, \quad |t| < \frac{1}{\sigma}$

notes Also known as the *Laplace* distribution.

Exponential(β)

pdf $f(x|\beta) = \frac{1}{\beta}e^{-x/\beta}, \quad 0 \leq x < \infty, \quad \beta > 0$

mean and variance $EX = \beta, \quad \text{Var } X = \beta^2$

mgf $M_X(t) = \frac{1}{1-\beta t}, \quad t < \frac{1}{\beta}$

notes Special case of the gamma distribution. Has the *memoryless* property. Has many special cases: $Y = X^{1/\gamma}$ is *Weibull*, $Y = \sqrt{2X/\beta}$ is *Rayleigh*, $Y = \alpha - \gamma \log(X/\beta)$ is *Gumbel*.

F

pdf $f(x|\nu_1, \nu_2) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{x^{\nu_1-2/2}}{(1+(\frac{\nu_1}{\nu_2})x)^{(\nu_1+\nu_2)/2}};$
 $0 \leq x < \infty; \quad \nu_1, \nu_2 = 1, \dots$

mean and variance $EX = \frac{\nu_2}{\nu_2-2}, \quad \nu_2 > 2,$

$\text{Var } X = 2 \left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}, \quad \nu_2 > 4$

moments (mgf does not exist) $EX^n = \frac{\Gamma(\frac{\nu_1+2n}{2})\Gamma(\frac{\nu_2-2n}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_2}{\nu_1}\right)^n, \quad n < \frac{\nu_2}{2}$

notes Related to chi squared ($F_{\nu_1, \nu_2} = \left(\frac{\chi_{\nu_1}^2}{\nu_1}\right) / \left(\frac{\chi_{\nu_2}^2}{\nu_2}\right)$, where the χ^2 's are independent) and t ($F_{1, \nu} = t_\nu^2$).

Gamma(α, β)

pdf $f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 \leq x < \infty, \quad \alpha, \beta > 0$

mean and variance $EX = \alpha\beta, \quad \text{Var } X = \alpha\beta^2$

mgf $M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}$

notes Some special cases are exponential ($\alpha = 1$) and chi squared ($\alpha = p/2$, $\beta = 2$). If $\alpha = \frac{3}{2}$, $Y = \sqrt{X/\beta}$ is *Maxwell*. $Y = 1/X$ has the *inverted gamma distribution*. Can also be related to the Poisson (Example 3.2.1).

Logistic(μ, β)

pdf $f(x|\mu, \beta) = \frac{1}{\beta} \frac{e^{-(x-\mu)/\beta}}{[1+e^{-(x-\mu)/\beta}]^2}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \beta > 0$

mean and variance $EX = \mu, \quad \text{Var } X = \frac{\pi^2\beta^2}{3}$

mgf $M_X(t) = e^{\mu t} \Gamma(1 - \beta t) \Gamma(1 + \beta t), \quad |t| < \frac{1}{\beta}$

notes The cdf is given by $F(x|\mu, \beta) = \frac{1}{1 + e^{-(x-\mu)/\beta}}$.

Lognormal(μ, σ^2)

pdf $f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \frac{e^{-(\log x - \mu)^2 / (2\sigma^2)}}{x}, \quad 0 \leq x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$

mean and variance $EX = e^{\mu + (\sigma^2/2)}, \quad \text{Var } X = e^{2(\mu + \sigma^2)} - e^{2\mu + \sigma^2}$

moments $EX^n = e^{n\mu + n^2\sigma^2/2}$
(mgf does not exist)

notes Example 2.3.5 gives another distribution with the same moments.

Normal(μ, σ^2)

pdf $f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2 / (2\sigma^2)}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$

mean and variance $EX = \mu, \quad \text{Var } X = \sigma^2$

mgf $M_X(t) = e^{\mu t + \sigma^2 t^2 / 2}$

notes Sometimes called the *Gaussian* distribution.

Pareto(α, β)

pdf $f(x|\alpha, \beta) = \frac{\beta \alpha^\beta}{x^{\beta+1}}, \quad a < x < \infty, \quad \alpha > 0, \quad \beta > 0$

mean and variance $EX = \frac{\beta \alpha}{\beta - 1}, \quad \beta > 1, \quad \text{Var } X = \frac{\beta \alpha^2}{(\beta - 1)^2 (\beta - 2)}, \quad \beta > 2$

mgf does not exist

t

pdf $f(x|\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}} \frac{1}{(1 + (\frac{x^2}{\nu}))^{(\nu+1)/2}}, \quad -\infty < x < \infty, \quad \nu = 1, \dots$

mean and variance $EX = 0, \quad \nu > 1, \quad \text{Var } X = \frac{\nu}{\nu - 2}, \quad \nu > 2$

moments $EX^n = \frac{\Gamma(\frac{\nu+1}{2}) \Gamma(\frac{\nu-n}{2})}{\sqrt{\pi} \Gamma(\frac{\nu}{2})} \nu^{n/2}$ if $n < \nu$ and even,
(mgf does not exist) $EX^n = 0$ if $n < \nu$ and odd.

notes Related to F ($F_{1,\nu} = t_\nu^2$).

Uniform(a, b)

pdf $f(x|a, b) = \frac{1}{b-a}, \quad a \leq x \leq b$

mean and variance $EX = \frac{b+a}{2}, \quad \text{Var } X = \frac{(b-a)^2}{12}$

mgf $M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$

notes If $a = 0$ and $b = 1$, this is a special case of the beta ($\alpha = \beta = 1$).

Weibull(γ, β)

pdf $f(x|\gamma, \beta) = \frac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta}, \quad 0 \leq x < \infty, \quad \gamma > 0, \quad \beta > 0$

mean and variance $EX = \beta^{1/\gamma} \Gamma\left(1 + \frac{1}{\gamma}\right), \quad \text{Var } X = \beta^{2/\gamma} \left[\Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma^2\left(1 + \frac{1}{\gamma}\right) \right]$

moments $EX^n = \beta^{n/\gamma} \Gamma\left(1 + \frac{n}{\gamma}\right)$

notes The mgf exists only for $\gamma \geq 1$. Its form is not very useful. A special case is exponential ($\gamma = 1$).
