We need these operators to have similar properties to the raising- and lowering operators for $Z$

\[ S^+_x |\uparrow_x \rangle = 0, \quad S^-_x |\uparrow_x \rangle = h |\uparrow_x \rangle, \quad S^+_x |\downarrow_x \rangle = h |\downarrow_x \rangle, \quad S^-_x |\downarrow_x \rangle = 0 \]

\[ S^+_x : \quad S^+_x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad 1 \rightarrow_x = (\frac{1}{\sqrt{2}}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[ S^+_x |\uparrow_x \rangle = 0 \Rightarrow (a \quad b) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ \Rightarrow a + b = 0, \quad c + d = 0 \Rightarrow a = -b, \quad c = -d \]

\[ S^+_x |\downarrow_x \rangle = \frac{1}{\sqrt{2}} |\uparrow_x \rangle \Rightarrow (c \quad -a) \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} (1 \quad 1) \]

\[ \Rightarrow a + a = h, \quad c + c = h \Rightarrow a = c = \frac{h}{2} \]

\[ \Rightarrow S^+_x = \frac{h}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]

\[ S^-_x : \quad S^-_x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad 1 \rightarrow_x = (\frac{1}{\sqrt{2}}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[ S^-_x |\downarrow_x \rangle = 0 \Rightarrow (a \quad b) \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a - b = 0 \quad c - d = 0 \]

\[ \Rightarrow a = b, \quad c = d \]

\[ S^-_x |\uparrow_x \rangle = h |\downarrow_x \rangle \Rightarrow (c \quad -a) \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} (1 \quad -1) \]

\[ \Rightarrow a + a = h, \quad c + c = -h \Rightarrow a = \frac{h}{2}, \quad c = -\frac{h}{2} \]

\[ \Rightarrow S^-_x = \frac{h}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \]