Problem 12.86

You have a bucket containing an unknown liquid. You also have a cube-shaped wooden block that you measure to be 8.0 cm on a side, but you don't know the mass or density of the block. To find the density of the liquid, you perform an experiment. First you place the wooden block in the liquid and measure the height of the top of the floating block above the liquid surface. Then you stack various numbers of U.S. quarter-dollar coins onto the block and measure the new value of $h$. The straight line that gives the best fit to the data you have collected is shown in the figure. The mass of one quarter is 5.670 g. Use this information and the slope and intercept of the straight-line fit to your data. The lowest point on the graph corresponds to 1.2 cm.

Part A

Calculate the density of the liquid (in kg/m³).

Express your answer using two significant figures.

**ANSWER:**

\[ \rho = 1200 \text{ kg/m}^3 \]

Correct

Part B

Calculate the mass of the block (in kg).

Express your answer using two significant figures.

**ANSWER:**

\[ m = 0.39 \text{ kg} \]

Correct

Gravitational Acceleration inside a Planet
Consider a spherical planet of uniform density $\rho$. The distance from the planet's center to its surface (i.e., the planet's radius) is $R_p$. An object is located a distance $R$ from the center of the planet, where $R < R_p$. (The object is located inside of the planet.)

**Part A**

Find an expression for the magnitude of the acceleration due to gravity, $g(R)$, inside the planet.

**Express the acceleration due to gravity in terms of $\rho$, $R$, $\pi$, and $G$, the universal gravitational constant.**

**Hint 1. Force due to planet's mass outside radius $R$**

From Newton's *Principia*, Proposition LXX, Theorem XXX:

*If to every point of a spherical surface there tend equal centripetal forces decreasing as the square of the distances from those points, I say, that a corpuscle placed within that surface will not be attracted by those forces any way.*

In other words, you don't have to worry about the portion of the planet's mass that is located outside of the radius $R$. The net gravitational force from this "outer shell" of mass will equal zero. You only have to worry about that portion of the planet's mass that is located within a radius $R$.

**Hint 2. Find the force on an object at distance $R$**

Suppose the object has a mass $m$. Find the magnitude of the gravitational force acting on this object when it is located a distance $R$ from the center of the planet.

**Express the force in terms of $m$, $\rho$, $\pi$, $R$, and $G$, the universal gravitational constant.**

**Hint 1. Find the mass within a radius $R$**

Find the net mass of the planet located within the radius $R$. Remember, the volume of a sphere is $(4/3)\pi R^3$.

**Express your answer in terms of $\rho$, $\pi$, and $R$.**

**ANSWER:**

$$M(R) = \frac{4}{3} \pi R^3 \rho$$

**Hint 2. Magnitude of gravitational force**

The general equation for the magnitude of the gravitational force is $F_{grav} = GMm/R^2$.

**ANSWER:**

$$F(R) = \frac{4}{3} \pi Gm\rho R$$

**Hint 3. Finding $g(R)$ from $F(R)$**

According to Newton's 2nd law, the net force acting on an object is given by $F_{net} = ma$. In this problem, $a = g(R)$ and $F_{net} = F(R)$ since the only force acting on the object is the gravitational force. Therefore, $g(R) = F(R)/m$, where $F(R)$ is the force you found in the previous hint.

Note that in this usage, both $F(R)$ and $a$ are magnitudes and hence are positive. By convention, $g$ (or $g(R)$ in this case) is the magnitude of the gravitational field. This gravitational field is a vector, with direction downward.

**ANSWER:**
Part B

Rewrite your result for \( g(R) \) in terms of \( g_p \), the gravitational acceleration at the surface of the planet, times a function of \( R \).

Express your answer in terms of \( g_p \), \( R \), and \( R_p \).

**Hint 1. Acceleration at the surface**

Note that the acceleration at the surface should be equal to the value of the function \( g(R) \) from Part A evaluated at the radius of the planet: \( g_p = g(R_p) \).

**ANSWER:**

\[
g(R) = g_p \frac{R}{R_p}
\]

**Correct**

Notice that \( g \) increases linearly with \( R \), rather than being proportional to \( 1/R^2 \). This assures that it is zero at the center of the planet, as required by symmetry.

Part C

Find a numerical value for \( \rho_{\text{earth}} \), the average density of the earth in kilograms per cubic meter. Use 6378 km for the radius of the earth, \( G = 6.67 \times 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2) \), and a value of \( g \) at the surface of 9.80 m/s^2.

Express your answer to three significant figures.

**Hint 1. How to approach the problem**

You already derived the relation needed to solve this problem in Part A:

\[
g(R) = (4/3)\pi G \rho R.
\]

At what distance \( R \) is \( g(R) \) known so that you could use this relation to find \( \rho \)?

**ANSWER:**

\[
\rho_{\text{earth}} = 5500 \text{ kg/m}^3
\]

**Correct**

± Escape Velocity

Learning Goal:
To introduce you to the concept of escape velocity for a rocket.

The escape velocity is defined to be the minimum speed with which an object of mass $m$ must move to escape from the gravitational attraction of a much larger body, such as a planet of total mass $M$. The escape velocity is a function of the distance of the object from the center of the planet $R$, but unless otherwise specified this distance is taken to be the radius of the planet because it addresses the question “How fast does my rocket have to go to escape from the surface of the planet?”

Part A

The key to making a concise mathematical definition of escape velocity is to consider the energy. If an object is launched at its escape velocity, what is the total mechanical energy $E_{\text{total}}$ of the object at a very large (i.e., infinite) distance from the planet? Follow the usual convention and take the gravitational potential energy to be zero at very large distances.

**Hint 1. Consider various cases**

If the object is given some speed less than its escape velocity, it will eventually return to the planet. If it is given some speed greater than its escape velocity, then even at very large distances, it will still have some velocity directed away from the planet. If the object is given its escape velocity exactly, its velocity at a very large distance from the planet will tend to zero (so, its kinetic energy will go to zero).

**ANSWER:**

$$E_{\text{total}} = 0$$

Correct

Consider the motion of an object between a point close to the planet and a point very very far from the planet. Indicate whether the following statements are true or false.

Part B

Angular momentum about the center of the planet is conserved.

**ANSWER:**

- true
- false

Correct

Part C

Total mechanical energy is conserved.

**ANSWER:**

- true
- false

Correct
Part D
Kinetic energy is conserved.
ANSWER:
- true
- false
Correct

Part E
The angular momentum about the center of the planet and the total mechanical energy will be conserved regardless of whether the object moves from small $R$ to large $R$ (like a rocket being launched) or from large $R$ to small $R$ (like a comet approaching the earth).
ANSWER:
- true
- false
Correct
What if the object is not moving directly away from or toward the planet but instead is moving at an angle $\theta$ from the normal? In this case, it will have a tangential velocity $v_{\text{tan}} = v \sin \theta$ and angular momentum $L = m v_{\text{tan}} R$. Since angular momentum is conserved, $v_{\text{tan}} = L / (m R)$ for any $R$, so $v_{\text{tan}}$ will go to 0 as $R$ goes to infinity. This means that angular momentum can be conserved without adding any kinetic energy at $R = \infty$. The important aspect for determining the escape velocity will therefore be the conservation of total mechanical energy.

Part F
Find the escape velocity $v_e$ for an object of mass $m$ that is initially at a distance $R$ from the center of a planet of mass $M$. Assume that $R \geq R_{\text{planet}}$, the radius of the planet, and ignore air resistance.
Express the escape velocity in terms of $R$, $M$, $m$, and $G$, the universal gravitational constant.

Hint 1. Determine the gravitational potential energy
Find $U$, the gravitational potential energy of the object at a distance $R$ from the center of the planet, with the gravitational potential energy taken to be zero when the object is infinitely far away from the planet.
Express your answer in terms of $R$, $M$, $m$, and $G$, the universal gravitational constant.
ANSWER:

$$U = \frac{-GMm}{R}$$

Hint 2. Determine the kinetic energy
Suppose that the object is given its escape velocity $v_e$. Find the initial kinetic energy $K$ of the object when it is at its initial distance $R$ from the center of the planet.
Express your answer in terms of $v_e$ and other given quantities.
Hint 3. Putting it all together

You can assume that no nonconservative forces act on the object as it moves through space. Hence, using conservation of mechanical energy, you can equate the object's initial mechanical energy (when it is at a distance \( R \) from the planet) to the object's mechanical energy at very large distances from the planet. This should allow you to solve for \( v_e \).

\[
K = \frac{1}{2} mv_e^2
\]

ANSWER:

\[
v_e = \sqrt{\frac{2GM}{R}}
\]

Correct

Does it surprise you that the escape velocity does not depend on the mass of the object? Even more surprising is that it does not depend on the direction (as long as the trajectory misses the planet). Any angular momentum given at radius \( R \) can be conserved with a tangential velocity that vanishes as \( R \) goes to infinity, so the angle at which the object is launched does not have a significant effect on the energy at large \( R \).

Exercise 13.24

The International Space Station makes 15.65 revolutions per day in its orbit around the earth.

Part A

Assuming a circular orbit, how high is this satellite above the surface of the earth?

ANSWER:

\[
h = 370 \text{ km}
\]

Correct

Kepler's 3rd Law

A planet moves in an elliptical orbit around the sun. The mass of the sun is \( M_s \). The minimum and maximum distances of the planet from the sun are \( R_1 \) and \( R_2 \), respectively.

Part A

Using Kepler's 3rd law and Newton's law of universal gravitation, find the period of revolution \( P \) of the planet as it moves around the sun. Assume that the mass of the planet is much smaller than the mass of the sun.

Use \( G \) for the gravitational constant.

Express the period in terms of \( G, M_s, R_1, \) and \( R_2 \).

Hint 1. Kepler's 3rd law
Kepler’s 3rd law states that the square of the period of revolution of a planet around the sun is proportional to the cube of the semi-major axis of its orbit. Try finding the period of a circular orbit and then using Kepler’s 3rd law (which applies equally to circular and elliptical orbits) to extend your result to an elliptical orbit.

**Hint 2. Find the semi-major axis**
Find the semi-major axis $a$.

**Express the semi-major axis in terms of $R_1$ and $R_2$.**

**Hint 1. Definition of semi-major axis**
The semi-major axis of an ellipse is half of its major axis. The sun is at the focus of the elliptical orbit and the focus lies on the major axis.

**ANSWER:**

\[
a = \frac{R_1 + R_2}{2}
\]

**Hint 3. Find the period of a circular orbit**
Find the period $P$ of a planet in a circular orbit of semi-major axis $a$.

**Express the period in terms of $a$, $M_s$, and $G$.**

**Hint 1. Formula for the period**
The period is $2\pi r / v$, where $r$ is the radius of the orbit and $v$ is the speed of the object. Note that this is the distance traveled in one orbit divided by the speed.

**Hint 2. Find the velocity**
Find the velocity $v$ of an object in an orbit of radius $r$ by setting the magnitude of the centripetal acceleration $a_{\text{cent}} = v^2 / r$ equal to the magnitude of the acceleration due to gravity.

**Express your answer in terms of $r$, $M_s$, and $G$.**

**ANSWER:**

\[
v = \sqrt{\frac{M_s G}{r}}
\]

**Hint 3. Radius of the orbit**
For a circle, the semi-major axis is just the radius.

**ANSWER:**

\[
P = 2\pi \sqrt{\frac{a^3}{GM_s}}
\]
Exercise 13.32

A uniform, spherical, 1800.0 kg shell has a radius of 5.00 m.

Part A

Find the gravitational force this shell exerts on a 1.50 kg point mass placed at the distance 5.02 m from the center of the shell.

ANSWER:

\[ F = 7.15 \times 10^{-9} \text{ N} \]

Correct

Part B

Find the gravitational force this shell exerts on a 1.50 kg point mass placed at the distance 4.98 m from the center of the shell.

ANSWER:

\[ F = 0 \text{ N} \]

Correct

Part C

Find the gravitational force this shell exerts on a 1.50 kg point mass placed at the distance 2.71 m from the center of the shell.

ANSWER:

\[ F = 0 \text{ N} \]

Correct

Alternative Exercise 13.98

Part A

\[ P = 2\sqrt{\frac{\pi^2}{2} \left( \frac{R_2 + R_1}{2} \right)^3 G M_s} \]
Calculate the percent difference between your weight in Sacramento, near sea level, and at the top of Mount Everest, which is 8800 m above sea level.

Express your answer using two significant figures.

ANSWER:

\[
\frac{\Delta w}{w} = 0.28 \% \\
\]

Correct

Exercise 13.22

On July 15, 2004, NASA launched the Aura spacecraft to study the earth's climate and atmosphere. This satellite was injected into an orbit 705 km above the earth's surface, and we shall assume a circular orbit.

Part A

How many hours does it take this satellite to make one orbit?

ANSWER:

\[ T = 1.65 \text{ hours} \]

Correct

Part B

How fast (in km/s) is the Aura spacecraft moving?

ANSWER:

\[ v = 7.49 \text{ km/s} \]

Correct

Score Summary:
Your score on this assignment is 99.7%.
You received 7.97 out of a possible total of 8 points.