Introduction to Capacitance

**Learning Goal:** To understand the meaning of capacitance and ways of calculating capacitance

When a positive charge $q$ is placed on a conductor that is insulated from ground, an electric field emanates from the conductor to ground, and the conductor will have a nonzero potential $V$ relative to ground. If more charge is placed on the conductor, this voltage will increase proportionately. The ratio of charge to voltage is called the capacitance $C$ of this conductor: $C = \frac{q}{V}$.

Capacitance is one of the central concepts in electrostatics, and specially constructed devices called capacitors are essential elements of electronic circuits. In a capacitor, a second conducting surface is placed near the first (they are often called electrodes), and the relevant voltage is the voltage between these two electrodes.

This tutorial is designed to help you understand capacitance by assisting you in calculating the capacitance of a parallel-plate capacitor, which consists of two plates each of area $A$ separated by a small distance $d$ with air or vacuum in between. In figuring out the capacitance of this configuration of conductors, it is important to keep in mind that the voltage difference is the line integral of the electric field between the plates.

**Part A**

What property of objects is best measured by their capacitance?

**ANSWER:**
- ability to conduct electric current
- ability to distort an external electrostatic field
- ability to store charge
- ability to store electrostatic energy

Capacitance is a measure of the ability of a system of two conductors to store electric charge and energy. Capacitance is defined as $C = \frac{Q}{V}$. This ratio remains constant as long as the system retains its geometry and the amount of dielectric does not change. Capacitors are special devices designed to combine a large capacitance with a small size. However, any pair of conductors separated by a dielectric (or vacuum) has some capacitance. Even an isolated electrode has a small capacitance. That is, if a charge $q$ is placed on it, its potential $V$ with respect to ground will change, and the ratio $\frac{q}{V}$ is its capacitance $C$.

**Part B**
Assume that charge \(-q\) is placed on the top plate, and \(+q\) is placed on the bottom plate. What is the magnitude of the electric field \(E\) between the plates?

**Part B.1 How do you find the magnitude of the electric field?**

*Part not displayed*

**Part B.2 What is the electric flux integral due to the electric field?**

*Part not displayed*

**Part B.3 Find the enclosed charge**

*Part not displayed*

**Hint B.4 Recall Gauss's law**

*Hint not displayed*

Express \(E\) in terms of \(q\) and other quantities given in the introduction, in addition to \(\varepsilon_0\) and any other constants needed.

**ANSWER:** \(E = \frac{q}{(\varepsilon_0 A)}\)

**Part C**

What is the voltage \(V\) between the plates of the capacitor?

**Hint C.1 The electric field is the derivative of the potential**

The voltage difference is the integral of the electric field from one plate to the other; in symbols, \(V = E \cdot d\).

Express \(V\) in terms of the quantities given in the introduction and any required physical constants.

**ANSWER:** \(V = \frac{q}{(\varepsilon_0 A)} \cdot d\)

**Part D**

Now find the capacitance \(C\) of the parallel-plate capacitor.

Express \(C\) in terms of quantities given in the introduction and constants like \(\varepsilon_0\).

**ANSWER:** \(C = \frac{q}{(\varepsilon_0 A) \cdot d}\)

You have derived the general formula for the capacitance of a parallel-plate capacitor with plate area \(A\) and plate separation \(d\). It is worth remembering.

**Part E**

Consider an air-filled charged capacitor. How can its capacitance be increased?

**Hint E.1 What does capacitance depend on?**

*Hint not displayed*

**ANSWER:**
- Increase the charge on the capacitor.
- Decrease the charge on the capacitor.
- Increase the spacing between the plates of the capacitor.
- Decrease the spacing between the plates of the capacitor.
- Increase the length of the wires leading to the capacitor plates.

**Part F**
Consider a charged parallel-plate capacitor. Which combination of changes would quadruple its capacitance?

**ANSWER:**
- Double the charge and double the plate area.
- Double the charge and double the plate separation.
- Halve the charge and double the plate separation.
- Halve the charge and double the plate area.
- Halve the plate separation; double the plate area.
- Double the plate separation; halve the plate area.

### Capacitance: A Review

**Learning Goal:** To review the meaning of capacitance and ways of changing the capacitance of a parallel-plate capacitor.

Capacitance is one of the central concepts in electrostatics. Understanding its meaning and the difference between its definition and the ways of calculating capacitance can be challenging at first. This tutorial is meant to help you become more comfortable with capacitance. Recall the fundamental formula for capacitance:

\[ C = \frac{Q}{V}, \]

where \( C \) is the capacitance in farads, \( Q \) is the charge stored on the plates in coulombs, and \( V \) is the potential difference (or voltage) between the plates. In the following problems it may help to keep in mind that the voltage is related to the strength of the electric field \( E \) and the distance between the plates, \( d \), by

\[ V = Ed. \]

**Part A**

What property of objects is best measured by their capacitance?

**ANSWER:** *Answer not displayed*

**Part B**

Consider an air-filled charged capacitor. How can its capacitance be increased?

**Hint B.1 What does capacitance depend on?**

*Hint not displayed*

**ANSWER:** *Answer not displayed*

**Part C**

Consider a charged parallel-plate capacitor. How can its capacitance be halved?

- A. Double the charge.
- B. Double the plate area.
- C. Double the plate separation.
- D. Halve the charge.
- E. Halve the plate area.
- F. Halve the plate separation.

Enter the letters of the correct actions in alphabetical order. For example, if actions A and B would achieve the desired result enter AB.

**ANSWER:** *Answer not displayed*

**Part D**
Consider a charged parallel-plate capacitor. Which combination of changes would quadruple its capacitance?

ANSWER: Answer not displayed

Capacitors in Parallel and Series

**Capacitors in Parallel**

**Learning Goal:** To understand how to calculate capacitance, voltage, and charge for a parallel combination of capacitors.

Frequently, several capacitors are connected together to form a collection of capacitors. We may be interested in determining the overall capacitance of such a collection. The simplest configuration to analyze involves capacitors connected in series or in parallel. More complicated setups can often (though not always!) be treated by combining the rules for these two cases. Consider the example of a parallel combination of capacitors: Three capacitors are connected to each other and to a battery as shown in the figure. The individual capacitances are \(C_1\), \(2C\), and \(3C\), and the battery's voltage is \(V\).

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**Part A**

If the potential of plate 1 is \(V\), then, in equilibrium, what are the potentials of plates 3 and 6? Assume that the negative terminal of the battery is at zero potential.

**Hint A.1 Electrostatic equilibrium**

**ANSWER:** Answer not displayed

**Part B**

If the charge of the first capacitor (the one with capacitance \(C\)) is \(Q\), then what are the charges of the second and third capacitors?

**Hint B.1 Definition of capacitance**

**Hint not displayed**

**ANSWER:** Answer not displayed

**Hint B.2 Voltages across the capacitors**

**Hint not displayed**

**ANSWER:** Answer not displayed

**Part C**

Suppose we consider the system of the three capacitors as a single "equivalent" capacitor. Given the charges of the three individual capacitors calculated in the previous part, find the total charge \(Q_{\text{eq}}\) for this equivalent capacitor.

Express your answer in terms of \(V\) and \(C\).
Part D
Using the value of $q_{wb}$, find the equivalent capacitance $C_{eq}$ for this combination of capacitors.

**Hint D.1 Using the definition of capacitance**

Express your answer in terms of $C$.  

**ANSWER:** $C_{eq} = \text{Answer not displayed}$

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### Capacitors in Series

**Learning Goal:** To understand how to calculate capacitance, voltage, and charge for a combination of capacitors connected in series.

Consider the combination of capacitors shown in the figure. Three capacitors are connected to each other in series, and then to the battery. The values of the capacitances are $C$, $2C$, and $3C$, and the applied voltage is $V$. Initially, all of the capacitors are completely discharged; after the battery is connected, the charge on plate 1 is $Q$.

**Part A**
What are the charges on plates 3 and 6?

**Hint A.1 Charges on capacitors connected in series**

**Hint A.2 The charges on a capacitor’s plates**

**ANSWER:** $\text{Answer not displayed}$

**Part B**
If the voltage across the first capacitor (the one with capacitance $C$) is $V$, then what are the voltages across the second and third capacitors?

**Hint B.1 Definition of capacitance**

**Hint B.2 Charges on the capacitors**

**ANSWER:** $\text{Answer not displayed}$

**Part C**
Find the voltage $V_1$ across the first capacitor.

**Hint C.1 How to analyze voltages**

Express your answer in terms of $V$.

**ANSWER:** $V_1 = \text{Answer not displayed}$

**Part D**
Find the charge $Q$ on the first capacitor.

Express your answer in terms of $C$ and $V$.

**ANSWER:** $Q = \text{Answer not displayed}$

**Part E**
Using the value of $Q$ just calculated, find the equivalent capacitance $C_{eq}$ for this combination of capacitors in series.

**Hint E.1 Using the definition of capacitance**

Express your answer in terms of $C$.

**ANSWER:** $C_{eq} = \text{Answer not displayed}$

### Finding the Capacitance

A parallel-plate capacitor is filled with a dielectric whose dielectric constant is $K$, increasing its capacitance from $C_1$ to $KC_1$. A second capacitor with capacitance $C_2$ is then connected in series with the first, reducing the net capacitance back to $C_1$.

**Part A**
What is the capacitance $C_2$ of the second capacitor?

**Part A.1 What do capacitors in series have in common?**

**Part not displayed**

**Part A.2 Find the net capacitance**

**Part not displayed**

Express your answer in terms of $K$, $C_1$, and constants.

**ANSWER:** $C_2 = \frac{(KC_1)}{(K-1)}$

As a check that this result makes sense, note that if $K \gg 1$, which corresponds to the first capacitor being replaced by a conductor, then $C_2 \approx C_1$. In the limit as $K$ approaches 1 (but is still greater than 1), $C_2$ becomes very large, effectively making the second capacitor a conductor. This is also expected, since if $K = 1$, then no change was made to the first capacitor.

### Properties of Capacitors

**Force between Capacitor Plates**
Consider a parallel-plate capacitor with plates of area $A$ and with separation $d$.

**Part A**
Find $F/V$, the magnitude of the force each plate experiences due to the other plate as a function of $V$, the potential drop across the capacitor.

### Part A.1 How to approach the problem
The plates of the capacitor exert forces on each other because one has a positive charge $+q$, and the other has a negative charge $-q$.

Which of the following equations is not applicable to this problem?

**ANSWER:**
- $F = qE$
- $V = \frac{q}{d}$
- $q = CV$
- $C = \varepsilon_0\frac{A}{d}$

### Part A.2 Method 1: Use the stored energy to find the force
What is $U$, the energy stored in the capacitor?

**Hint A.2.a Which formula to use**

**Hint not displayed**

Express your answer in terms of $V$, $A$, $d$, and $\varepsilon_0$.

**ANSWER:**
$$U = \frac{1}{2} \cdot \frac{q^2}{d} \cdot \frac{A}{V}$$

If the plate separation were changed while the voltage was kept constant, the stored energy would change. The force between the plates would be the quantity that would be multiplied by the change in the plate separation to obtain the change in energy. In other words, $F = \frac{dU}{dd}$.

### Part A.3 Method 2: Use the product of the charge and the field to find the force
The plates of the capacitor exert forces on each other because one has a positive charge $+q$, and the other has a negative charge $-q$. Find $E_{\text{total}}$, the magnitude of the electric field between the plates of the capacitor due to a single plate.

**Hint A.3.a How to approach the problem**
From the symmetry of the situation, you can conclude that the field due to any one plate is half the total field in the region between the plates. Then the force is the charge on a plate times the electric field due to the other plate.

Note: Since the plate is flat and infinite, it cannot exert a force on itself.

### Part A.3.b The relationship between voltage and electric field
We can approximate a parallel-plate capacitor by assuming that the electric field $E_{\text{total}}$ points directly from one plate to the other and has a constant magnitude. In this case, what is the relationship between the voltage of the capacitor $V$, the electric field, and the separation between the plates $d$?

Express your answer in terms of some or all of the variables $E_{\text{total}}$ and $d$.

**ANSWER:**
$$V = E_{\text{total}}d$$

Express your answer in terms of any or all of $V$, $A$, $d$, and $\varepsilon_0$.
Both plates make the same contribution $E_{\text{plate}} = \frac{V}{d}$ in the same direction. Therefore, the total field $E_{\text{total}}$ is equal to $2E_{\text{plate}} = \frac{V}{d}$, which is the more familiar result.

Express your answer in terms of given quantities and $\varepsilon_0$.

**ANSWER:** $V = \frac{\varepsilon_0 \cdot A \cdot V^2}{\varepsilon_0}$

**The Capacitor as an Energy-Storing Device**

**Learning Goal:** To understand that the charge stored by capacitors represents energy; to be able to calculate the stored energy and its changes under different circumstances.

An air-filled parallel-plate capacitor has plate area $A$ and plate separation $d$. The capacitor is connected to a battery that creates a constant voltage $V$.

**Part A**

Find the energy $U_0$ stored in the capacitor.

**Hint A.1 Formula for the energy of a capacitor**

$E = \frac{1}{2} CV^2$

Express your answer in terms of $A$, $d$, $V$, and $\varepsilon_0$. Remember to enter $\varepsilon_0$ as $\varepsilon_0$.

**ANSWER:** $U_0 = Answer not displayed$

**Part B**

The capacitor is now disconnected from the battery, and the plates of the capacitor are then slowly pulled apart until the separation reaches $3d$. Find the new energy $U_1$ of the capacitor after this process.

**Part B.1 What quantity remains constant?**

**Part not displayed**

**Part B.2 Find the charge on the capacitor**

**Part not displayed**

**Part B.3 How does the capacitance change?**

**Part not displayed**

**Part B.4 What is the formula for the energy?**

**Part not displayed**

Express your answer in terms of $A$, $d$, $V$, and $\varepsilon_0$.

**ANSWER:** $U_1 = Answer not displayed$

**Part C**

The capacitor is now reconnected to the battery, and the plate separation is restored to $d$. A dielectric plate is slowly moved into the capacitor until the entire space between the plates is filled. Find the energy $U_2$ of the dielectric-filled capacitor. The capacitor remains connected to the battery. The dielectric constant is $\varepsilon_r$.

Express your answer in terms of $A$, $d$, $V$, $\varepsilon_r$, and $\varepsilon_0$.

**ANSWER:** $U_2 = Answer not displayed$
**Capacitors with Partial Dielectrics**

Consider two parallel-plate capacitors identical in shape, one aligned so that the plates are horizontal, and the other with the plates vertical.

**Part A**

The horizontal capacitor is filled halfway with a material that has dielectric constant $\varepsilon_r$. What fraction $f$ of the area of the vertical capacitor should be filled (as shown in the figure) with the same dielectric so that the two capacitors have equal capacitance?

**Part A.1 Capacitance of a parallel-plate capacitor filled with air**

The capacitance of a capacitor depends solely on its geometry. What is the capacitance $C$ of a capacitor with area $A$ and separation $d$?

Express your answer in terms of $\varepsilon_r$, $A$, and $d$.

**ANSWER:**

$C = \varepsilon_r A / d$

**Hint A.2 Effect of a Dielectric**

Inserting a dielectric material between the plates of a capacitor decreases the electric field between the plates, because the molecules in the dielectric align themselves like the electrons in a conductor. The net result of inserting the dielectric is an increase in the capacitor's capacitance by a factor of the dielectric constant: $C_{\text{new}} = K C_{\text{air}}$.

**Part A.3 Modeling the horizontal capacitor**

The horizontal capacitor can be modeled as two smaller capacitors: one with half the separation completely filled with air, and one with half the separation completely filled with the dielectric. These two smaller capacitors are connected in series. What is the capacitance $C_h$ of the horizontal capacitor, with original area $A$ and separation $d$?

**Hint A.3.a Connecting capacitors in series**

Express your answer in terms of $A$, $K$, $\varepsilon_r$, and $\varepsilon_0$.

**ANSWER:**

$C_h = \left(2K\varepsilon_r A / d\right) / (K + 3)$

As a check that this all makes sense, note that if $K = 1$, then $C_h = C - \varepsilon_0 A / d$ (the capacitance with no dielectric), and if $K \gg 1$, then $C_h \approx 2\varepsilon_0 A / d$ (half of the capacitor is a conductor, and the net capacitance is that of a capacitor without a dielectric but a separation $d/2$).

**Part A.4 Modeling the vertical capacitor**
Capacitor with Partial Dielectric

Consider a parallel-plate capacitor that is partially filled with a dielectric of dielectric constant $K$. The dielectric has the same same height as the separation of the plates of the capacitor but fills a fraction $f$ of the area of the capacitor. The capacitance of the capacitor when the dielectric is completely removed is $C_0$.

Part A
What is the capacitance $C(f)$ of this capacitor as a function of $f$?

Part A.1 Modeling the partly filled capacitor
- Part not displayed

Part A.2 Find the capacitance of the air-filled portion
- Part not displayed

Part A.3 Find the capacitance of the dielectric-filled portion
- Part not displayed

Hint A.4 Special cases
- Hint not displayed

Express $C(f)$ in terms of $C_0$, $f$, and $K$.

ANSWER: $C(f) = \text{Answer not displayed}$

Energy of a Capacitor in the Presence of a Dielectric

An dielectric-filled parallel-plate capacitor has plate area $A$ and plate separation $d$. The capacitor is connected to a battery that creates a constant voltage $V$. The dielectric constant is $K$.

Part A
Find the energy $U_f$ of the dielectric-filled capacitor. The capacitor remains connected to the battery.

Part A.1 What is the energy of a capacitor?
- Part not displayed

Part A.2 Find the capacitance
- Part not displayed

Express your answer in terms of $A$, $d$, $V$, $K$, and $f_0$.

ANSWER: $U_f = \text{Answer not displayed}$

Part B
The dielectric plate is now slowly pulled out of the capacitor, which remains connected to the battery. Find the energy $U_f$ of the capacitor at the moment when the capacitor is half filled with the dielectric.

Hint B.1 What quantity remains constant?
A parallel-plate vacuum capacitor is connected to a battery and charged until the stored electric energy is $U$. The battery is removed, and then a dielectric material with dielectric constant $K$ is inserted into the capacitor, filling the space between the plates. Finally, the capacitor is fully discharged through a resistor (a resistor is connected across the capacitor terminals).

Part A
Part B

Consider the same situation as in the previous part, except that the charging battery remains connected while the dielectric is inserted. The battery is then disconnected and the capacitor discharged. For this situation, what is \( u_i \), the energy dissipated in the resistor?

**Hint B.1 The energy of a capacitor**

*Hint not displayed*

**Hint B.2 The effects of the dielectric**

*Hint not displayed*

Express your answer in terms of \( u_i \) and other given quantities.

**ANSWER:**

\[ u_i = \frac{K}{U} \]

Can you explain where the extra energy came from? (Answer below)

In this case, the extra energy comes from the battery which provides some extra charge, to maintain the potential difference across the plates.

The first case is a little more involved. The dielectric experiences an attractive force as it is inserted into the capacitor. Therefore it would either gain kinetic energy, or work would have to be done against any resistive forces on it that prevent it from gaining speed. This is where the energy goes. You could actually calculate the force on the dielectric as \( F = \frac{dU(x)}{dx} \), where \( U(x) \) is the energy in the capacitor as a function of the length \( x \) of the dielectric inserted into the capacitor.