Changes in Lengths of Axially Loaded Members

**Problem 2.2-1** The T-shaped arm $ABC$ shown in the figure lies in a vertical plane and pivots about a horizontal pin at $A$. The arm has constant cross-sectional area and total weight $W$. A vertical spring of stiffness $k$ supports the arm at point $B$.

Obtain a formula for the elongation $\delta$ of the spring due to the weight of the arm.

**Solution 2.2-1** T-shaped arm

**FREE-BODY DIAGRAM OF ARM**

$F = $ tensile force in the spring

$\Sigma M_A = 0 \implies$

$F(b) - \frac{W}{3}\left(\frac{b}{2}\right) - \frac{W}{3}\left(\frac{3b}{2}\right) - \frac{W}{3}(2b) = 0$

$F = \frac{4W}{3}$

$\delta =$ elongation of the spring

$\delta = \frac{F}{k} = \frac{4W}{3k}$

---

**Problem 2.2-2** A steel cable with nominal diameter 25 mm (see Table 2-1) is used in a construction yard to lift a bridge section weighing 38 kN, as shown in the figure. The cable has an effective modulus of elasticity $E = 140$ GPa.

(a) If the cable is 14 m long, how much will it stretch when the load is picked up?

(b) If the cable is rated for a maximum load of 70 kN, what is the factor of safety with respect to failure of the cable?
Solution 2.2-2  Bridge section lifted by a cable

(a) STRETCH OF CABLE
\[ \delta = \frac{WL}{EA} = \frac{(38 \text{ kN})(14 \text{ m})}{(140 \text{ GPa})(304 \text{ mm}^2)} \]
\[ = 12.5 \text{ mm} \]

(b) FACTOR OF SAFETY
\[ \frac{P_{ULT}}{P_{max}} = \frac{406 \text{ kN}}{70 \text{ kN}} = 5.8 \]

Problem 2.2-3  A steel wire and a copper wire have equal lengths and support equal loads \( P \) (see figure). The moduli of elasticity for the steel and copper are \( E_s = 30,000 \text{ ksi} \) and \( E_c = 18,000 \text{ ksi} \), respectively.

(a) If the wires have the same diameters, what is the ratio of the elongation of the copper wire to the elongation of the steel wire?

(b) If the wires stretch the same amount, what is the ratio of the diameter of the copper wire to the diameter of the steel wire?

Solution 2.2-3  Steel wire and copper wire

(a) RATIO OF ELONGATIONS (EQUAL Diameters)
\[ \delta_c = \delta_s \]
\[ \frac{E_c}{E_s} \]
\[ \frac{P}{A_c} = \frac{P}{A_s} \]
\[ \delta_c = \frac{P}{E_c A_c} = \frac{P}{E_s A_s} \]
\[ \frac{\delta_c}{\delta_s} = \frac{E_s}{E_c} \]
\[ = \frac{30}{18} = 1.67 \]

(b) RATIO OF DIAMETERS (EQUAL ELONGATIONS)
\[ \delta_c = \delta_s \]
\[ \frac{E_c}{E_s} \]
\[ \frac{P}{A_c} = \frac{P}{A_s} \]
\[ E_s \left( \frac{\pi d_c^2}{4} \right) = E_s \left( \frac{\pi d_s^2}{4} \right) \]
\[ \frac{E_s}{E_c} \]
\[ \frac{d_c^2}{d_s^2} = \sqrt{\frac{E_s}{E_c}} = \sqrt{\frac{30}{18}} = 1.29 \]
Problem 2.2-4  By what distance $h$ does the cage shown in the figure move downward when the weight $W$ is placed inside it?

Consider only the effects of the stretching of the cable, which has axial rigidity $EA = 10,700$ kN. The pulley at $A$ has diameter $d_A = 300$ mm and the pulley at $B$ has diameter $d_B = 150$ mm. Also, the distance $L_1 = 4.6$ m, the distance $L_2 = 10.5$ m, and the weight $W = 22$ kN. (*Note:* When calculating the length of the cable, include the parts of the cable that go around the pulleys at $A$ and $B$.)

**Solution 2.2-4  Cage supported by a cable**

\[ L_1 = 4.6 \text{ m} \]
\[ L_2 = 10.5 \text{ m} \]
\[ EA = 10,700 \text{ kN} \]
\[ W = 22 \text{ kN} \]

**LENGTH OF CABLE**

\[ L = L_1 + 2L_2 + \frac{1}{4}(\pi d_A) + \frac{1}{2}(\pi d_B) \]
\[ = 4,600 \text{ mm} + 21,000 \text{ mm} + 236 \text{ mm} + 236 \text{ mm} \]
\[ = 26,072 \text{ mm} \]

**ELONGATION OF CABLE**

\[ \delta = \frac{TL}{EA} = \frac{(11 \text{ kN})(26,072 \text{ mm})}{(10,700 \text{ kN})} = 26.8 \text{ mm} \]

**LOWERING OF THE CAGE**

\[ h = \text{distance the cage moves downward} \]
\[ h = \frac{1}{2} \delta = 13.4 \text{ mm} \]

Problem 2.2-5  A safety valve on the top of a tank containing steam under pressure $p$ has a discharge hole of diameter $d$ (see figure). The valve is designed to release the steam when the pressure reaches the value $p_{\text{max}}$.

If the natural length of the spring is $L$ and its stiffness is $k$, what should be the dimension $h$ of the valve? (Express your result as a formula for $h$.)
Solution 2.2-5  Safety valve

\[ h = \text{height of valve (compressed length of the spring)} \]
\[ d = \text{diameter of discharge hole} \]
\[ p = \text{pressure in tank} \]

\[ p_{\text{max}} = \text{pressure when valve opens} \]
\[ L = \text{natural length of spring} \ (L > h) \]
\[ k = \text{stiffness of spring} \]

**FORCE IN COMpressed SPRING**
\[ F = k(L - h) \quad \text{(From Eq. 2-1a)} \]

**PRESSURE FORCE ON SPRING**
\[ P = p_{\text{max}} \left( \frac{\pi d^2}{4} \right) \]

**EQUATE FORCES AND SOLVE FOR \( h \):**
\[ F = P \quad k(L - h) = \frac{p_{\text{max}} d^2}{4} \]
\[ h = L - \frac{p_{\text{max}} d^2}{4k} \]

---

Problem 2.2-6  The device shown in the figure consists of a pointer \( ABC \) supported by a spring of stiffness \( k = 800 \text{ N/m} \). The spring is positioned at distance \( b = 150 \text{ mm} \) from the pinned end \( A \) of the pointer. The device is adjusted so that when there is no load \( P \), the pointer reads zero on the angular scale.

If the load \( P = 8 \text{ N} \), at what distance \( x \) should the load be placed so that the pointer will read 3° on the scale?

---

Solution 2.2-6  Pointer supported by a spring

**FREE-BODY DIAGRAM OF POINTER**

\[ \sum M_A = 0 \quad \Leftrightarrow \]
\[ -P_x + (k\delta)b = 0 \quad \text{or} \quad \delta = \frac{P_x}{kb} \]

Let \( \alpha \) = angle of rotation of pointer
\[ \tan \alpha = \frac{\delta}{b} = \frac{P_x}{kb^2} \quad x = \frac{kb^2}{P} \tan \alpha \]

**SUBSTITUTE NUMERICAL VALUES:**
\[ \alpha = 3^\circ \]
\[ x = \frac{(800 \text{ N/m})(150 \text{ mm})^2}{8 \text{ N}} \tan 3^\circ \]
\[ = 118 \text{ mm} \]
Problem 2.2-7  Two rigid bars, \( AB \) and \( CD \), rest on a smooth horizontal surface (see figure). Bar \( AB \) is pivoted end \( A \) and bar \( CD \) is pivoted at end \( D \). The bars are connected to each other by two linearly elastic springs of stiffness \( k \). Before the load \( P \) is applied, the lengths of the springs are such that the bars are parallel and the springs are without stress.

Derive a formula for the displacement \( \delta_c \) at point \( C \) when the load \( P \) is acting. (Assume that the bars rotate through very small angles under the action of the load \( P \).)

Solution 2.2-7  Two bars connected by springs

\( k \) = stiffness of springs
\( \delta_c \) = displacement at point \( C \) due to load \( P \)

**Free-body diagrams**

\( F_1 \) = tensile force in first spring
\( F_2 \) = compressive force in second spring

**Equilibrium**

\[
\sum M_A = 0 \quad -bF_1 + 2bF_2 = 0 \quad F_1 = 2F_2
\]

\[\sum M_D = 0 \quad 2bP - 2bF_1 + bF_2 = 0 \quad F_2 = 2F_1 - 2P\]

Solving, \( F_1 = \frac{4P}{3} \quad F_2 = \frac{2P}{3} \)

\( \delta_B \) = displacement of point \( B \)
\( \delta_C \) = displacement at point \( C \)
\( \Delta_1 \) = elongation of first spring

\[\Delta_1 = \delta_c - \frac{\delta_B}{2}\]

\( \Delta_2 \) = shortening of second spring

\[\Delta_2 = \delta_B - \frac{\delta_C}{2}\]

Also, \( \Delta_1 = \frac{F_1}{k} = \frac{4P}{3k} \quad \Delta_2 = \frac{F_2}{k} = \frac{2P}{3k} \)

**Solve the equations:**

\[\Delta_1 = \Delta_1 \quad \delta_c - \frac{\delta_B}{2} = \frac{4P}{3k}\]

\[\Delta_2 = \Delta_2 \quad \delta_B - \frac{\delta_C}{2} = \frac{2P}{3k}\]

Eliminate \( \delta_B \) and obtain \( \delta_c \):

\[\delta_c = \frac{20P}{9k}\]
Problem 2.2-8  The three-bar truss $ABC$ shown in the figure has a span $L = 3$ m and is constructed of steel pipes having cross-sectional area $A = 3900$ mm$^2$ and modulus of elasticity $E = 200$ GPa. A load $P$ acts horizontally to the right at joint $C$.

(a) If $P = 650$ kN, what is the horizontal displacement of joint $B$?
(b) What is the maximum permissible load $P_{\text{max}}$ if the displacement of joint $B$ is limited to 1.5 mm?

---

Solution 2.2-8  Truss with horizontal load

From force triangle,

$$ F_{AB} = \frac{P}{2} \text{ (tension)} $$

(a) **HORIZONTAL DISPLACEMENT $\delta_B$**

$$ P = 650 \text{ kN} $$

$$ \delta_B = \frac{F_{AB} L_{AB}}{EA} = \frac{PL}{2EA} $$

$$ = \frac{(650 \text{ kN})(3 \text{ m})}{2(200 \text{ GPa})(3900 \text{ mm}^2)} $$

$$ = 1.25 \text{ mm} \quad \leftarrow $$

(b) **MAXIMUM LOAD $P_{\text{max}}$**

$$ \delta_{\text{max}} = 1.5 \text{ mm} $$

$$ \frac{P_{\text{max}}}{\delta_{\text{max}}} = \frac{P}{\delta} \quad P_{\text{max}} = P \left( \frac{\delta_{\text{max}}}{\delta} \right) $$

$$ P_{\text{max}} = (650 \text{ kN}) \left( \frac{1.5 \text{ mm}}{1.25 \text{ mm}} \right) $$

$$ = 780 \text{ kN} \quad \leftarrow $$
Problem 2.2-9 An aluminum wire having a diameter \( d = 2 \text{ mm} \) and length \( L = 3.8 \text{ m} \) is subjected to a tensile load \( P \) (see figure). The aluminum has modulus of elasticity \( E = 75 \text{ GPa} \).

If the maximum permissible elongation of the wire is 3.0 mm and the allowable stress in tension is 60 MPa, what is the allowable load \( P_{\text{max}} \)?

Solution 2.2-9 Aluminum wire in tension

\[
\begin{align*}
\text{=} & = \frac{EA}{L} \delta_{\text{max}} \\
\text{=} & = \frac{(75 \text{ GPa})(3.142 \text{ mm}^2)}{3.8 \text{ m}} (3.0 \text{ mm}) \\
\text{=} & = 186 \text{ N}
\end{align*}
\]

**Maximum load based upon stress**

\[
\sigma_{\text{allow}} = 60 \text{ MPa} \quad \sigma = \frac{P}{A}
\]

\[
P_{\text{max}} = A\sigma_{\text{allow}} = (3.142 \text{ mm}^2)(60 \text{ MPa})
\]

\[= 189 \text{ N}\]

**Allowable load**

Elongation governs. \( P_{\text{max}} = 186 \text{ N} \)

Problem 2.2-10 A uniform bar \( AB \) of weight \( W = 25 \text{ N} \) is supported by two springs, as shown in the figure. The spring on the left has stiffness \( k_1 = 300 \text{ N/m} \) and natural length \( L_1 = 250 \text{ mm} \). The corresponding quantities for the spring on the right are \( k_2 = 400 \text{ N/m} \) and \( L_2 = 200 \text{ mm} \). The distance between the springs is \( L = 350 \text{ mm} \), and the spring on the right is suspended from a support that is distance \( h = 80 \text{ mm} \) below the point of support for the spring on the left.

At what distance \( x \) from the left-hand spring should a load \( P = 18 \text{ N} \) be placed in order to bring the bar to a horizontal position?
Solution 2.2-10  Bar supported by two springs

\[ \Sigma M_A = 0 \quad \therefore \]
\[ F_1 L - P x - \frac{W L}{2} = 0 \]  \hspace{1cm} (Eq. 1)
\[ \Sigma F_{\text{vert}} = 0 \quad \uparrow \quad \downarrow \]
\[ F_1 + F_2 - P - W = 0 \]  \hspace{1cm} (Eq. 2)

SOLVE Eqs. (1) AND (2):

\[ F_1 = P \left(1 - \frac{x}{L}\right) + \frac{W}{2} \]
\[ F_2 = \frac{P x}{L} + \frac{W}{2} \]

SUBSTITUTE NUMERICAL VALUES:

UNITS: Newtons and meters

\[ F_1 = (18) \left(1 - \frac{x}{0.350}\right) + 12.5 = 30.5 - 51.429x \]
\[ F_2 = (18) \left(\frac{x}{0.350}\right) + 12.5 = 51.429x + 12.5 \]

ELONGATIONS OF THE SPRINGS

\[ \delta_1 = \frac{F_1}{k_1} = \frac{F_1}{300} = 0.10167 - 0.17143x \]
\[ \delta_2 = \frac{F_2}{k_2} = \frac{F_2}{400} = 0.12857x + 0.031250 \]

BAR AB REMAINS HORIZONTAL

Points A and B are the same distance below the reference line (see figure above).

\[ \therefore \quad L_1 + \delta_1 = h + L_2 + \delta_2 \]
or \[ 0.250 + 0.10167 - 0.17143x \]
\[ = 0.080 + 0.200 + 0.12857x + 0.031250 \]

SOLVE FOR x:

\[ 0.300 x = 0.040420 \quad x = 0.1347 \text{ m} \]
\[ x = 135 \text{ mm} \quad \leftarrow \]
Problem 2.2-11  A hollow, circular, steel column \((E = 30,000 \text{ ksi})\) is subjected to a compressive load \(P\), as shown in the figure. The column has length \(L = 8.0 \text{ ft}\) and outside diameter \(d = 7.5 \text{ in}\). The load \(P = 85 \text{ k}\).

If the allowable compressive stress is 7000 psi and the allowable shortening of the column is 0.02 in., what is the minimum required wall thickness \(t_{\min}\)?

Solution 2.2-11  Column in compression

\[ P = 85 \text{ k} \]
\[ E = 30,000 \text{ ksi} \]
\[ L = 8.0 \text{ ft} \]
\[ d = 7.5 \text{ in.} \]
\[ \sigma_{\text{allow}} = 7,000 \text{ psi} \]
\[ \delta_{\text{allow}} = 0.02 \text{ in.} \]

**Required area based upon allowable stress**

\[ \sigma = \frac{P}{A} \quad A = \frac{P}{\sigma_{\text{allow}}} = \frac{85 \text{ k}}{7,000 \text{ psi}} = 12.14 \text{ in}^2 \]

**Required area based upon allowable shortening**

\[ \delta = \frac{PL}{EA} \quad A = \frac{PL}{E\delta_{\text{allow}}} = \frac{(85 \text{ k})(96 \text{ in.})}{(30,000 \text{ ksi})(0.02 \text{ in.})} = 13.60 \text{ in}^2 \]

Shortening governs

\[ A_{\text{min}} = 13.60 \text{ in}^2 \]

**Minimum thickness \(t_{\min}\)**

\[ A = \frac{\pi}{4}[d^2 - (d - 2t)^2] \quad \text{or} \quad \frac{4A}{\pi} - d^2 = -(d - 2t)^2 \]

\[ (d - 2t)^2 = d^2 - \frac{4A}{\pi} \text{ or } d - 2t = \sqrt{d^2 - \frac{4A}{\pi}} \]

\[ t = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \frac{A}{\pi}} \quad \text{or} \quad t_{\min} = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \frac{A_{\text{min}}}{\pi}} \]

**Substitute numerical values**

\[ t_{\min} = \frac{7.5 \text{ in.}}{2} - \sqrt{\left(\frac{7.5 \text{ in.}}{2}\right)^2 - \frac{13.60 \text{ in}^2}{\pi}} \]

\[ t_{\min} = 0.63 \text{ in.} \]
**Problem 2.2-12** The horizontal rigid beam ABCD is supported by vertical bars BE and CF and is loaded by vertical forces $P_1 = 400 \text{ kN}$ and $P_2 = 360 \text{ kN}$ acting at points A and D, respectively (see figure). Bars BE and CF are made of steel ($E = 200 \text{ GPa}$) and have cross-sectional areas $A_{BE} = 11,100 \text{ mm}^2$ and $A_{CF} = 9,280 \text{ mm}^2$. The distances between various points on the bars are shown in the figure.

Determine the vertical displacements $\delta_A$ and $\delta_D$ of points A and D, respectively.

**Solution 2.2-12** Rigid beam supported by vertical bars

**Free-body diagram of bar ABCD**

\[ \begin{align*}
\sum M_B &= 0 \quad \Leftrightarrow \quad (400 \text{ kN})(1.5 \text{ m}) + F_{CF}(1.5 \text{ m}) - (360 \text{ kN})(3.6 \text{ m}) = 0 \\
F_{CF} &= 464 \text{ kN} \\
\sum M_C &= 0 \quad \Leftrightarrow \quad (400 \text{ kN})(3.0 \text{ m}) - F_{BE}(1.5 \text{ m}) - (360 \text{ kN})(2.1 \text{ m}) = 0 \\
F_{BE} &= 296 \text{ kN}
\end{align*} \]

**Shortening of bar BE**

\[ \delta_{BE} = \frac{F_{BE} L_{BE}}{EA_{BE}} = \frac{(296 \text{ kN})(3.0 \text{ m})}{(200 \text{ GPa})(11,100 \text{ mm}^2)} = 0.400 \text{ mm} \]

**Shortening of bar CF**

\[ \delta_{CF} = \frac{F_{CF} L_{CF}}{EA_{CF}} = \frac{(464 \text{ kN})(2.4 \text{ m})}{(200 \text{ GPa})(9,280 \text{ mm}^2)} = 0.600 \text{ mm} \]

**Displacement diagram**

\[ \begin{align*}
\delta_{BE} - \delta_A &= \delta_{CF} - \delta_{BE} \quad \text{or} \quad \delta_A = 2\delta_{BE} - \delta_{CF} \\
\delta_A &= 2(0.400 \text{ mm}) - 0.600 \text{ m} \quad \text{(Downward)} \\
\delta_D - \delta_{CF} &= \frac{2.1}{1.5} \delta_{CF} - \delta_{BE} \\
\delta_D &= \frac{12}{5} \delta_{CF} - \frac{7}{5} \delta_{BE} \\
&= \frac{12}{5}(0.600 \text{ mm}) - \frac{7}{5}(0.400 \text{ mm}) \\
&= 0.880 \text{ mm} \quad \text{(Downward)}
\end{align*} \]
Problem 2.2-13  A framework $ABC$ consists of two rigid bars $AB$ and $BC$, each having length $b$ (see the first part of the figure). The bars have pin connections at $A$, $B$, and $C$ and are joined by a spring of stiffness $k$. The spring is attached at the midpoints of the bars. The framework has a pin support at $A$ and a roller support at $C$, and the bars are at an angle $\alpha$ to the horizontal.

When a vertical load $P$ is applied at joint $B$ (see the second part of the figure) the roller support $C$ moves to the right, the spring is stretched, and the angle of the bars decreases from $\alpha$ to the angle $\theta$.

Determine the angle $\theta$ and the increase $\delta$ in the distance between points $A$ and $C$. (Use the following data; $b = 8.0$ in., $k = 16$ lb/in., $\alpha = 45^\circ$, and $P = 10$ lb.)

---

Solution 2.2-13  Framework with rigid bars and a spring

With no load

$\begin{align*}
L_1 &= \text{span from } A \text{ to } C \\
&= 2b \cos \alpha \\
S_1 &= \text{length of spring} \\
&= \frac{L_1}{2} = b \cos \alpha
\end{align*}$

With load $P$

$\begin{align*}
L_2 &= \text{span from } A \text{ to } C \\
&= 2b \cos \theta \\
S_2 &= \text{length of spring} \\
&= \frac{L_2}{2} = b \cos \theta
\end{align*}$

Free-body diagram of $BC$

$\begin{align*}
h &= \text{height from } C \text{ to } B = b \sin \theta \\
\frac{L_2}{2} &= b \cos \theta \\
F &= \text{force in spring due to load } P \\
\Sigma M_B = 0 \\
\frac{P}{2} \left( \frac{L_2}{2} \right) - F \left( \frac{h}{2} \right) &= 0 \text{ or } P \cos \theta = F \sin \theta \quad \text{(Eq. 1)}
\end{align*}$

(Continued)
DETERMINE THE ANGLE \( \theta \)

\[ \Delta S = \text{elongation of spring} \]

\[ = S_2 - S_1 = b(\cos \theta - \cos \alpha) \]

For the spring: \( F = k(\Delta S) \)

\[ F = bk(\cos \theta - \cos \alpha) \]

Substitute \( F \) into Eq. (1):

\[ P \cos \theta = bk(\cos \theta - \cos \alpha)(\sin \theta) \]

or \[ \frac{P}{bk} \cot \theta - \cos \theta + \cos \alpha = 0 \] (Eq. 2)

This equation must be solved numerically for the angle \( \theta \).

DETERMINE THE DISTANCE \( \delta \)

\[ \delta = L_2 - L_1 = 2b \cos \theta - 2b \cos \alpha \]

\[ = 2b(\cos \theta - \cos \alpha) \]

From Eq. (2): \[ \cos \alpha = \cos \theta - \frac{P \cot \theta}{bk} \]

Therefore,

\[ \delta = 2b \left( \cos \theta - \cos \alpha + \frac{P \cot \theta}{bk} \right) \]

\[ = \frac{2P}{b} \cot \theta \] (Eq. 3)

NUMERICAL RESULTS

\( b = 8.0 \text{ in.} \quad k = 16 \text{ lb/in.} \quad \alpha = 45^\circ \quad P = 10 \text{ lb} \)

Substitute into Eq. (2):

\[ 0.078125 \cot \theta - \cos \theta + 0.707107 = 0 \] (Eq. 4)

Solve Eq. (4) numerically:

\[ \theta = 35.1^\circ \] (Eq. 5)

Substitute into Eq. (3):

\[ \delta = 1.78 \text{ in.} \] (Eq. 6)

Problem 2.2-14  Solve the preceding problem for the following data:

\( b = 200 \text{ mm}, \quad k = 3.2 \text{ kN/m}, \quad \alpha = 45^\circ, \quad \text{and} \quad P = 50 \text{ N} \).

Solution 2.2-14  Framework with rigid bars and a spring

See the solution to the preceding problem.

Eq. (2): \[ \frac{P}{bk} \cot \theta - \cos \theta + \cos \alpha = 0 \]

Eq. (3): \[ \delta = \frac{2P}{k} \cot \theta \]

NUMERICAL RESULTS

\( b = 200 \text{ mm} \quad k = 3.2 \text{ kN/m} \quad \alpha = 45^\circ \quad P = 50 \text{ N} \)

Substitute into Eq. (2):

\[ 0.078125 \cot \theta - \cos \theta + 0.707107 = 0 \] (Eq. 4)

Solve Eq. (4) numerically:

\[ \theta = 35.1^\circ \] (Eq. 5)

Substitute into Eq. (3):

\[ \delta = 44.5 \text{ mm} \] (Eq. 6)
Changes in Lengths under Nonuniform Conditions

Problem 2.3-1  Calculate the elongation of a copper bar of solid circular cross section with tapered ends when it is stretched by axial loads of magnitude 3.0 k (see figure).

The length of the end segments is 20 in. and the length of the prismatic middle segment is 50 in. Also, the diameters at cross sections $A$, $B$, $C$, and $D$ are 0.5, 1.0, 1.0, and 0.5 in., respectively, and the modulus of elasticity is 18,000 ksi.  

(Hint: Use the result of Example 2-4.)

Solution 2.3-1  Bar with tapered ends

$$d_A = d_D = 0.5 \text{ in.} \quad P = 3.0 \text{ k}$$
$$d_B = d_C = 1.0 \text{ in.} \quad E = 18,000 \text{ ksi}$$

**END SEGMENT ($L = 20 \text{ in.}$)**

From Example 2-4:

$$
\delta = \frac{4PL}{\pi E d_A d_B} \\
\delta_1 = \frac{4(3.0 \text{ k})(20 \text{ in.})}{\pi(18,000 \text{ ksi})(0.5 \text{ in.})(1.0 \text{ in.})} = 0.008488 \text{ in.}
$$

**MIDDLE SEGMENT ($L = 50 \text{ in.}$)**

$$
\delta_2 = \frac{PL}{EA} = \frac{(3.0 \text{ k})(50 \text{ in.})}{(18,000 \text{ ksi})(\frac{1}{4})(1.0 \text{ in.})^2} = 0.01061 \text{ in.}
$$

**ELONGATION OF BAR**

$$
\delta = \sum NL = 2\delta_1 + \delta_2 \\
= 2(0.008488 \text{ in.}) + (0.01061 \text{ in.}) \\
= 0.0276 \text{ in.}
$$

Problem 2.3-2  A long, rectangular copper bar under a tensile load $P$ hangs from a pin that is supported by two steel posts (see figure). The copper bar has a length of 2.0 m, a cross-sectional area of 4800 mm$^2$, and a modulus of elasticity $E_c = 120$ GPa. Each steel post has a height of 0.5 m, a cross-sectional area of 4500 mm$^2$, and a modulus of elasticity $E_s = 200$ GPa.

(a) Determine the downward displacement $\delta$ of the lower end of the copper bar due to a load $P = 180$ kN.

(b) What is the maximum permissible load $P_{\text{max}}$ if the displacement $\delta$ is limited to 1.0 mm?
Solution 2.3-2  Copper bar with a tensile load

\[ L_c = 2.0 \text{ m} \]
\[ A_c = 4800 \text{ mm}^2 \]
\[ E_c = 120 \text{ GPa} \]
\[ L_s = 0.5 \text{ m} \]
\[ A_s = 4500 \text{ mm}^2 \]
\[ E_s = 200 \text{ GPa} \]

(a) DOWNWARD DISPLACEMENT \( \delta \) (\( P = 180 \text{ kN} \))

\[ \delta_i = \frac{PL_i}{E_i A_i} = \frac{(180 \text{ kN})(2.0 \text{ m})}{(120 \text{ GPa})(4800 \text{ mm}^2)} = 0.625 \text{ mm} \]
\[ \delta_s = \frac{P/2L_s}{E_s A_s} = \frac{(90 \text{ kN})(0.5 \text{ m})}{(200 \text{ GPa})(4500 \text{ mm}^2)} = 0.050 \text{ mm} \]
\[ \delta = \delta_i + \delta_s = 0.625 \text{ mm} + 0.050 \text{ mm} = 0.675 \text{ mm} \]

(b) MAXIMUM LOAD \( P_{\text{max}} \) (\( \delta_{\text{max}} = 1.0 \text{ mm} \))

\[ P_{\text{max}} = \frac{\delta_{\text{max}}}{\delta} P = P \left( \frac{\delta_{\text{max}}}{\delta} \right) \]
\[ P_{\text{max}} = (180 \text{ kN}) \left( \frac{1.0 \text{ mm}}{0.675 \text{ mm}} \right) = 267 \text{ kN} \]

Problem 2.3-3  A steel bar \( AD \) (see figure) has a cross-sectional area of 0.40 in.\(^2\) and is loaded by forces \( P_1 = 2700 \text{ lb}, P_2 = 1800 \text{ lb}, \) and \( P_3 = 1300 \text{ lb}. \) The lengths of the segments of the bar are \( a = 60 \text{ in.}, \)
\( b = 24 \text{ in.}, \) and \( c = 36 \text{ in.} \)

(a) Assuming that the modulus of elasticity \( E = 30 \times 10^6 \text{ psi}, \) calculate the change in length \( \delta \) of the bar. Does the bar elongate or shorten?

(b) By what amount \( P \) should the load \( P_3 \) be increased so that the bar does not change in length when the three loads are applied?

Solution 2.3-3  Steel bar loaded by three forces

\[ A = 0.40 \text{ in.}^2 \]
\[ P_1 = 2700 \text{ lb} \]
\[ P_2 = 1800 \text{ lb} \]
\[ P_3 = 1300 \text{ lb} \]
\[ E = 30 \times 10^6 \text{ psi} \]

AXIAL FORCES

\[ N_{AB} = P_1 + P_2 - P_3 = 3200 \text{ lb} \]
\[ N_{BC} = P_2 - P_3 = 500 \text{ lb} \]
\[ N_{CD} = -P_3 = -1300 \text{ lb} \]

\[ \delta = \sum \frac{N_i L_i}{E_i A_i} = \frac{1}{E A} (N_{AB} L_{AB} + N_{BC} L_{BC} + N_{CD} L_{CD}) \]
\[ = \frac{1}{(30 \times 10^6 \text{ psi})(0.40 \text{ in.}^2)} [(3200 \text{ lb})(60 \text{ in.}) + (500 \text{ lb})(24 \text{ in.}) - (1300 \text{ lb})(36 \text{ in.})] = 0.0131 \text{ in. (elongation)} \]
Problem 2.3-4  A rectangular bar of length \( L \) has a slot in the middle half of its length (see figure). The bar has width \( b \), thickness \( t \), and modulus of elasticity \( E \). The slot has width \( b/4 \).

(a) Obtain a formula for the elongation \( \delta \) of the bar due to the axial loads \( P \).

(b) Calculate the elongation of the bar if the material is high-strength steel, the axial stress in the middle region is 160 MPa, the length is 750 mm, and the modulus of elasticity is 210 GPa.

Solution 2.3-4  Bar with a slot

\[
\delta = \frac{PL}{6Ebt} \left( \frac{1}{4} + \frac{4}{6} + \frac{1}{4} \right) = \frac{7PL}{6Ebt}
\]

Stress in middle region

\[
\sigma = \frac{P}{A} = \frac{P}{(\frac{3}{4}bt)} \quad \text{or} \quad \frac{P}{bt} = \frac{3\sigma}{4}
\]

Substitute into the equation for \( \delta \):

\[
\delta = \frac{7PL}{6Ebt} = \frac{7L}{6E} \left( \frac{P}{bt} \right) = \frac{7L}{6E} \left( \frac{3\sigma}{4} \right) = \frac{7\sigma L}{8E}
\]

(b) Substitute numerical values:

\[
\sigma = 160 \text{ MPa} \quad L = 750 \text{ mm} \quad E = 210 \text{ GPa}
\]

\[
\delta = \frac{7(160 \text{ MPa})(750 \text{ mm})}{8 (210 \text{ GPa})} = 0.500 \text{ mm}
\]
Problem 2.3-5  Solve the preceding problem if the axial stress in the middle region is 24,000 psi, the length is 30 in., and the modulus of elasticity is $30 \times 10^6$ psi.

Solution 2.3-5  Bar with a slot

Stress in middle region

$$\sigma = \frac{P}{A} = \frac{P}{\left(\frac{1}{2} bt\right)} = \frac{4P}{3bt} \quad \text{or} \quad \frac{P}{bt} = \frac{3\sigma}{4}.$$ 

Substitute into the equation for $\delta$:

$$\delta = \frac{7PL}{6Ebt} = \frac{7L}{6E} \left(\frac{P}{bt}\right) = \frac{7L}{6E} \left(\frac{3\sigma}{4}\right) = \frac{7\sigma L}{8E}.$$

(b) Substitute numerical values:

$$\sigma = 24,000 \text{ psi} \quad L = 30 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$$

$$\delta = \frac{7(24,000 \text{ psi})(30 \text{ in.})}{8(30 \times 10^6 \text{ psi})} = 0.0210 \text{ in.}.$$

Problem 2.3-6  A two-story building has steel columns $AB$ in the first floor and $BC$ in the second floor, as shown in the figure. The roof load $P_1$ equals 400 kN and the second-floor load $P_2$ equals 720 kN. Each column has length $L = 3.75$ m. The cross-sectional areas of the first- and second-floor columns are 11,000 mm$^2$ and 3,900 mm$^2$, respectively.

(a) Assuming that $E = 206$ GPa, determine the total shortening $\delta_{AC}$ of the two columns due to the combined action of the loads $P_1$ and $P_2$.

(b) How much additional load $P_0$ can be placed at the top of the column (point $C$) if the total shortening $\delta_{AC}$ is not to exceed 4.0 mm?

Solution 2.3-6  Steel columns in a building

(a) Shortening $\delta_{AC}$ of the two columns

$$\delta_{AC} = \sum \frac{N_i L_i}{E_i A_i} = \frac{N_{AB} L}{EA_{AB}} + \frac{N_{BC} L}{EA_{BC}}$$

$$= \frac{(1120 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(11,000 \text{ mm}^2)} + \frac{(400 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(3,900 \text{ mm}^2)}$$

$$= 1.8535 \text{ mm} + 1.8671 \text{ mm} = 3.7206 \text{ mm}.$$
Problem 2.3-7  A steel bar 8.0 ft long has a circular cross section of diameter \( d_1 = 0.75 \) in. over one-half of its length and diameter \( d_2 = 0.50 \) in. over the other half (see figure). The modulus of elasticity \( E = 30 \times 10^6 \) psi.

(a) How much will the bar elongate under a tensile load \( P = 5000 \) lb?

(b) If the same volume of material is made into a bar of constant diameter \( d \) and length 8.0 ft, what will be the elongation under the same load \( P \)?

Solution 2.3-7  Bar in tension

Original bar:

- \( V_o = A_1 L + A_2 L = L(A_1 + A_2) \)

Prismatic bar: \( V_p = A_p (2L) \)

Equate volumes and solve for \( A_p \):

\[
V_o = V_p \quad L(A_1 + A_2) = A_p (2L)
\]

\[
A_p = \frac{A_1 + A_2}{2} = \frac{1}{2} \left( \frac{\pi}{4} (d_1^2 + d_2^2) \right)
\]

\[
= \frac{\pi}{8} \left[ (0.75 \text{ in.})^2 + (0.50 \text{ in.})^2 \right] = 0.3191 \text{ in.}^2
\]

\[
\delta = \frac{P(2L)}{EA_p} = \frac{(5000 \text{ lb})(2)(48 \text{ in.})}{(30 \times 10^6 \text{ psi})(0.3191 \text{ in.}^2)}
\]

\[
= 0.0501 \text{ in.} \quad \leftarrow
\]

Note: A prismatic bar of the same volume will always have a smaller change in length than will a nonprismatic bar, provided the constant axial load \( P \), modulus \( E \), and total length \( L \) are the same.