

Élie Cartan's Theory of Moving Frames

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The Story of Symmetry

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- Klein publishes his famous Erlangen program in 1872, characterizing all known non-Euclidean geometries as the study of various homogeneous spaces (a notion based on the continuous group concept.)
- Lie utilizes the infinitesimal part of the new theory (the modern Lie algebras) in a search for a “Galois Theory” of differential equations.
- His work centers around finding *invariants* of group actions and *symmetry* of the equations.
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- Lie and Klein did not have global notions such as manifolds.

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- In the hands of Élie Cartan (1910-1930) the 19th century finite dimensional transformation groups became the modern Lie groups, the objects that permeate so much of mathematics and physics.
- Also, Cartan created a technique to study geometric objects, “repère mobile” or moving frames.
- Unfortunately, reading Cartan’s papers is notoriously difficult so the technique was not developed further while the mathematics community struggled understanding Cartan.
- Finally, S.S. Chern (1940s), a collaborator of Cartan’s, popularized the method among geometers.
- Still, there was never a rigorous mathematical formulation of the method, and it was only known to work in special cases, until Olver and Fels provided it in 1997.

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The finite and the infinite

- Consider the rotation group $SO(2) = S^1$ acting on the plane \mathbb{R}^2 :

$$(x, u) \rightarrow (x \cos(\theta) - u \sin(\theta), x \sin(\theta) + u \cos(\theta)).$$

- Consider transformations on $M = \{(x, u) \in \mathbb{R}^2 \mid u \neq 0\}$, defined by

$$(x, u) \rightarrow (f(x), \frac{u}{f'(x)}),$$

for arbitrary local diffeomorphisms f . This example of an infinite dimensional pseudo-group of transformations was known to Sophus Lie. The prefix “pseudo” stems from the fact that composition of two such transformations may not be possible (the image of one must lie in the domain of the other.)

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The objects of study

A smooth (right) action of a Lie group G on a manifold M is a smooth map $\Psi : M \times G \rightarrow M$ such that

$$\Psi(x, gh) = \Psi(\Psi(x, g), h), \quad \forall x \in M, g, h \in G.$$

Some remarks:

- Whenever convenient, we shall simply write $x \cdot g$ instead of $\Psi(x, g)$. The above condition then reads $x \cdot (gh) = (x \cdot g) \cdot h$.
- For a fixed $g \in G$, the map $x \rightarrow x \cdot g$ has inverse $x \rightarrow x \cdot g^{-1}$.
- The set $\mathcal{O}_x := \{x \cdot g \mid g \in G\}$ is called the *orbit of G through $x \in M$* .
- A smooth map $f : M \rightarrow N$ is called *invariant under the action* if $f(x \cdot g) = f(x)$, $\forall x \in M, g \in G$. This is equivalent to f being constant on the orbits of G in M .

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Origin of the moving frame

Cartan developed his theory of moving frames for Lie group actions in an attempt to solve the *congruence problem*: Given two submanifolds $N, \bar{N} \subseteq M$, determine whether or not there is a group element, g , such that $\bar{N} = N \cdot g$.

The well known *Frénet-Serret frame* is the classical example of a moving frame. In this case, the group action is translations and rotations of curves in \mathbb{R}^3 . Two curves are congruent under the group action if and only if their Frénet-Serret frames are equivalent.

The key element to the early results on moving frames was the naturality of the exterior derivative

$$\varphi^* d = d\varphi^*.$$

The modern definition, due to Olver and Fels (1997), takes a different route.

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Jets and jet bundles

The most difficult conceptual construction in the theory of symmetry is the *jet bundle*, a fundamental concept in the differential geometric theory of partial differential equations.

We shall describe the *local coordinate structure* of jet bundles. This means considering a manifold of the type $M = X \times U \subseteq \mathbb{R}^{p+q}$, where $X \subseteq \mathbb{R}^p$ is interpreted as the space of *independent variables* and $U \subseteq \mathbb{R}^q$ the space of *dependent variables*.

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The simplest case

For convenience, let $p = q = 1$. The first jet space, $J^{(1)}(M)$ is the space $M \times \mathbb{R} = X \times U \times \mathbb{R}$, where the third coordinate is to be interpreted as the *first derivative*, and a general point in $J^{(1)}(M)$ is written (x, u, u_x) .

Analogously, the k th jet space is $J^{(k)}(M) = M \times \mathbb{R}^k$ and a general point has the form $(x, u, u_x, \dots, u_{kx})$, where we interpret u_{kx} as being $\frac{d^k}{dx^k} u$.

If G acts smoothly on $M = X \times U$, there is an *induced action* of G on $J^{(1)}(M)$, called the prolonged action, described as follows.

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Prolonged action

Consider $(x_0, u_0, u_{x_0}) \in J^{(1)}(M)$ and let $f : X \rightarrow U$ be a smooth function (possibly just defined on a neighborhood of x_0) whose graph, $\Gamma_f \subseteq M$, “passes through” the point (x_0, u_0, u_{x_0}) . That is, $f(x_0) = u_0$ and $f'(x_0) = u_{x_0}$.

For $g \in G$ close enough to the identity, the curve $\Gamma_f \cdot g$ will be the graph of a smooth function, \tilde{f} , on some neighborhood around \tilde{x}_0 , where $(\tilde{x}_0, \tilde{u}_0) := (x_0, u_0) \cdot g$. Define

$$\tilde{u}_{x_0} := \left. \frac{d}{d\tilde{x}} \right|_{\tilde{x}=\tilde{x}_0} \tilde{f}(\tilde{x}).$$

We define the prolonged action of G on $J^{(1)}(M)$ according to

$$(x_0, u_0, u_{x_0}) \cdot g := (\tilde{x}_0, \tilde{u}_0, \tilde{u}_{x_0}).$$

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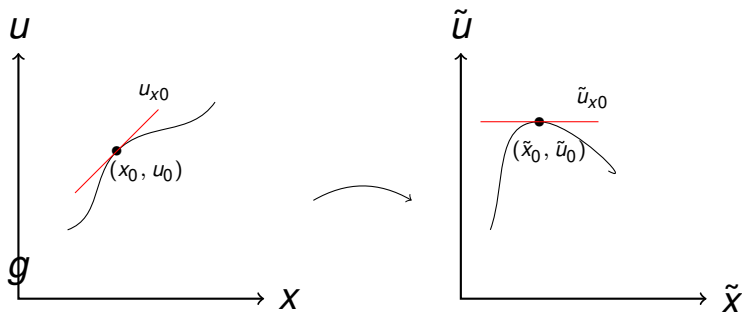
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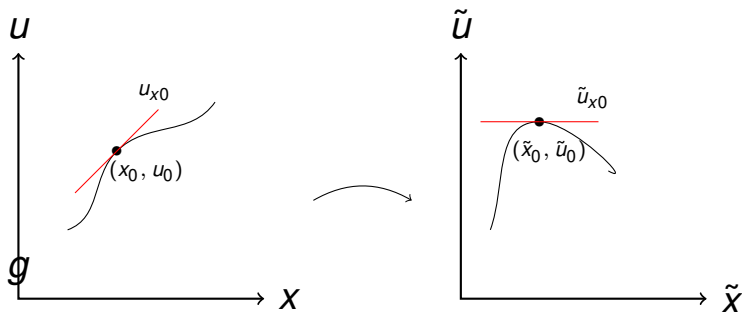
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Prolongation of $SO(2)$

The prolonged action of $SO(2)$ on \mathbb{R}^2 by rotations to the first and second order jet spaces is given by

$$(x, u) \rightarrow (x \cos(\theta) - u \sin(\theta), x \sin(\theta) + u \cos(\theta)),$$

$$u_x \rightarrow \frac{\sin(\theta) + u_x \cos(\theta)}{\cos(\theta) - u_x \sin(\theta)},$$

$$u_{xx} \rightarrow \frac{u_{xx}}{(\cos(\theta) - u_x \sin(\theta))^3}.$$

Equi-affine geometry of curves in \mathbb{R}^2

This geometry is governed by the action of the special affine group $SA(2) = SL(2) \times \mathbb{R}^2$ on \mathbb{R}^2 via

$$(x, u) \rightarrow (\delta(x-a) - \beta(u-b), -\gamma(x-a) + \alpha(u-b)), \quad \alpha\delta - \beta\gamma = 1.$$

The first few prolongations of this action are

$$u_x \rightarrow -\frac{\gamma - \alpha u_x}{\delta - \beta u_x}, \quad u_{xx} \rightarrow -\frac{u_{xx}}{(\delta - \beta u_x)^3},$$

$$u_{xxx} \rightarrow -\frac{(\delta - \beta u_x)u_{xxx} + 3\beta u_{xx}^2}{(\delta - \beta u_x)^5},$$

$$u_{xxxx} \rightarrow -\frac{(\delta - \beta u_x)^2 u_{xxxx} + 10\beta(\delta - \beta u_x)u_{xx}u_{xxx} + 15\beta^2 u_{xx}^3}{(\alpha + \beta u_x)^7}.$$

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Lie's "Galois Theory"

Let $\Delta(x, u^{(n)}) = 0$, $x \in X \subseteq \mathbb{R}^p$, $u \in U \subseteq \mathbb{R}^q$, be a differential equation of order n . A group, G , acting on the space $M = X \times U$, is said to be a *symmetry* of $\Delta(x, u^{(n)}) = 0$ if it moves graphs of (local) solutions to $\Delta = 0$ to other (local) solutions.

Given the symmetry group of $\Delta = 0$, it is sometimes possible to *reduce the order* of $\Delta = 0$. In favorable cases, a symmetry group of dimension r allows the order to be reduced by r .

When the order can be reduced to zero, in the classical terminology, $\Delta = 0$ is said to have been *solved by quadratures*. The term Galois Theory comes from Lie's attempt to classify those differential equations that can be solved in this way, in analogy to solvability of polynomials.

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The infinitesimal

Lie's great discovery was of infinitesimal methods for finding symmetry groups of differential equations.

His theorem says that a function $f : M \rightarrow N$ is invariant under the action of G on M ($f(x \cdot g) = f(x)$) if and only if $\mathbf{v}(f) = 0$ for all elements $\mathbf{v} \in \mathfrak{g}$, in the Lie algebra of G , where $\mathbf{v}(f)$ is the induced action of \mathfrak{g} on M .

The major advantage of this result is not having to deal with the complicated *non-linear* condition $f(x \cdot g) = f(x)$ and instead deal with the *linear* condition $\mathbf{v}(f) = 0$.

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The infinitesimal

Lie's great discovery was of infinitesimal methods for finding symmetry groups of differential equations.

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Reducing order

Consider the homogeneous ODE

$$u_x = F(x/u).$$

This equation has symmetry group \mathbb{R}_+ acting on $X \times U = \mathbb{R}^2$ by

$$(x, u) \cdot \lambda \rightarrow (x\lambda, u\lambda).$$

A process suggested by Lie, of “straightening out” the symmetry, gives rise to a change of variables

$$y = x/u, \quad w = \log(x),$$

but in these variables the original equation becomes

$$w_y = \frac{1}{F(y) - y},$$

an equation of order zero in w_y , which can be integrated (solved by quadratures.)

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N'other's Theorem

In the case of systems of *Euler-Lagrange equations* coming from a variational problem with functional $\int_{\Omega} L(x, u^{(n)})dx$, Lie's infinitesimal theory leads to the celebrated *N'other Theorem*, a fundamental result in the calculus of variations and physics.

The theorem says that there is a one-to-one correspondence between one dimensional symmetry groups of the functional and *conservation laws* of the system.

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Freeness and Regularity

The ultimate definition of a moving frame for Lie group actions requires us to restrict to group actions that are *free* and *regular*.

An action Ψ as above is free if $x \cdot g = x$ for some $x \in M$, $g \in G$, means that $g = e$, where e is the identity element of G .

Regularity of an action means roughly that the group orbits cannot behave too wildly. Regularity guarantees that the set of group orbits has a manifold structure.

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Prolonging until free

When confronted with an action that isn't free we prolong the action to a suitably high order jet bundle hoping it will eventually become free.

A result by Ovsiannikov says that for locally effective actions, the prolonged action will eventually become locally free (on a neighborhood of the identity.)

For locally free actions, by the inverse function theorem, the group orbits have the same dimension as the Lie group. In particular, we can have no hope of freeness if the manifold has strictly smaller dimension than the Lie group.

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Freeness of $SA(2)$ acting on \mathbb{R}^2

In this case, we have a five dimensional Lie group, $SA(2)$, acting on a two dimensional manifold \mathbb{R}^2 .

For there to be any hope of freeness, we will at least have to prolong this action to the five dimensional space $J^{(3)}(\mathbb{R}^2) = \mathbb{R}^5$. We already demonstrated the prolonged action on the third order jet bundle.

To check for local freeness, Olver has given a condition based on the *Lie determinant*, similar to the Wronskian for linear dependence of functions. The action of $SA(2)$ turns out to be free on $J^{(3)}$.

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The Moving Frame

Here is the definition of moving frame, due to Olver and Fels.

Definition

A (left) moving frame for a free and regular action of a Lie group G , on a manifold M , is a (left) equivariant map $\rho : M \rightarrow G$.

Left equivariance means that $\rho(x \cdot g) = g^{-1} \cdot \rho(x)$.

Do not be alarmed if you fail to see the connection of this definition with geometrical objects such as the Frénet-Serret frame.

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Invariants

For a general group action, it can be very difficult finding the invariant functions satisfying $f(x \cdot g) = f(x)$. Even using Lie's simplifying linear condition, $\mathbf{v}(f) = 0$, finding f requires using the method of characteristics to solve a first order PDE.

However, a moving frame makes this task trivial!

Indeed, the function $I : M \rightarrow M$, taking $x \rightarrow x \cdot \rho(x)$ gives a complete list of invariants of the action, in that any other invariant is a function of I . The powerful *Frobenius theorem* from differential geometry guarantees this.

To see that I is invariant, simply note that

$$I(x \cdot g) = (x \cdot g) \cdot \rho(x \cdot g) = x \cdot g \cdot g^{-1} \cdot \rho(x) = x \cdot \rho(x) = I(x).$$

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Constructing a moving frame

The construction of a moving frame for a free and regular group action on an m dimensional manifold with $r \leq m$ dimensional orbits, depends on a choice of a local cross-section to the group orbits.

A local cross section to the orbits means an $m - r$ dimensional submanifold, \mathcal{K} , that intersects each orbit exactly once in some neighborhood in the manifold.

We can then define a moving frame $\rho(x) \in G$ to be the unique (by freeness) element of G that moves x onto the cross section,

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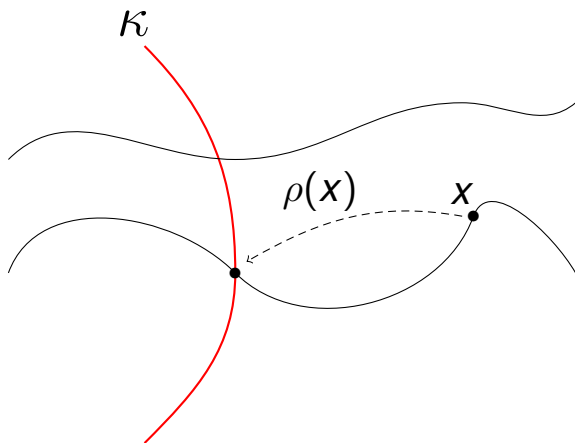
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Cross section and a moving frame



Equivariance

It is easy to see that this procedure gives a local equivariant map $\rho : M \rightarrow G$. Indeed, the unique element of G that moves $x \cdot g$ into \mathcal{K} is $g^{-1} \cdot \rho(x)$,

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Invariantization

For any smooth function, $F : M \rightarrow N$, defined on the domain of our moving frame, we can *invariantize* it, defining “a version” of F , $\iota(F)$, such that $\iota(F)(x)$ is the value of F at the cross section, where the orbit through x intersects it.

This means defining the invariantization of F to be the function

$$\iota(F)(x) := F(x \cdot \rho(x)).$$

Notice that we first calculate $F(x \cdot g)$ and then *plug in* the formula for the moving frame, i.e. putting $g = \rho(x)$.

This invariantization process can be generalized to other geometrical objects, where we essentially *pull back* the object of interest to the cross section, thereby making them invariant.

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Invariantizing in the cotangent bundle

On the domain of definition of the moving frame map $\rho : M \rightarrow G$, we can invariantize differential forms on M . Skipping the theoretical details, invariantizing the local coordinate one forms dx^i is particularly easy in practise.

We simply calculate the differential of the action on M , given by $x \rightarrow x \cdot g$, and then replace g by $\rho(x)$ in the outcome.

Invariantizing any smooth basis for the cotangent bundle, we obtain a G -invariant coframe, $\varpi_1, \dots, \varpi_m$.

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Example: $SO(2) \ltimes \mathbb{R}^2$ acting on \mathbb{R}^2

Expanding on our running example of $SO(2)$, the semi direct product $SO(2) \ltimes \mathbb{R}^2 = SE(2)$ acts on \mathbb{R}^2 by rotations and translations,

$$(x, u) \rightarrow (x \cos(\theta) - u \sin(\theta) + a, x \sin(\theta) + u \cos(\theta) + b),$$

where $a, b \in \mathbb{R}$. Within the Erlangen program, this homogeneous space represents “high-school geometry”.

$SE(2)$ is a three dimensional Lie group and prolonging to the first jet space of \mathbb{R}^2 , $J^{(1)}(\mathbb{R}^2) = \mathbb{R}^3$ its action becomes free.

In $J^{(1)}(\mathbb{R}^2)$, with coordinates (x, u, u_x) , the action in the third variable is

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The group orbits in $J^{(1)}(\mathbb{R}^2)$ are three dimensional, so any cross section has to be zero dimensional. Let the point $\mathcal{K} = \{x = u = u_x = 0\}$ be our cross section.

To construct a moving frame for this cross section, for each point $(x, u, u_x) \in J^{(1)}(\mathbb{R}^2)$, we need to determine a $g \in SE(2)$ such that $(x, u, u_x) \cdot g \in \mathcal{K}$. This means solving the following equations for θ , a and b .

$$0 = x \cos(\theta) - u \sin(\theta) + a,$$

$$0 = x \sin(\theta) + u \cos(\theta) + b,$$

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A moving frame

The solution to this system of equations is

$$\theta = -\tan^{-1}(u_x) \quad a = x, \quad b = u,$$

and so a moving frame $\rho : J^{(1)} \rightarrow SE(2)$ is given by

$$(x, u, u_x) \rightarrow (-\theta, a, b) = (\tan^{-1}(u_x), x, u).$$

Invariantizing

Prolonging to $J^{(2)}$, we have an action on the extra variable u_{xx} ,

$$u_{xx} \rightarrow \frac{u_{xx}}{(\cos(\theta) - u_x \sin(\theta))^3}.$$

The cross section \mathcal{K} as a subset in $J^{(2)}$ is now a one dimensional submanifold (the u_{xx} axis.)

The three dimensional group $SE(2)$ is acting freely on the four dimensional space $J^{(2)} = \mathbb{R}^4$ and the Frobenius theorem tells us that there is exactly $4-3=1$ invariant function found in the components of $z \rightarrow z \cdot \rho(z)$, where $z = (x, u, u_x, u_{xx})$.

Indeed, using the moving frame ρ , constructed above, we find that this function sends (x, u, u_x, u_{xx}) to

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The three dimensional group $SE(2)$ is acting freely on the four dimensional space $J^{(2)} = \mathbb{R}^4$ and the Frobenius theorem tells us that there is exactly $4-3=1$ invariant function found in the components of $z \rightarrow z \cdot \rho(z)$, where $z = (x, u, u_x, u_{xx})$.

Indeed, using the moving frame ρ , constructed above, we find that this function sends (x, u, u_x, u_{xx}) to

$$(0, 0, 0, \frac{u_{xx}}{(1 + u_x^2)^{3/2}}).$$

Curvature

We see that at the second order (on $J^{(2)}$) there is just one fundamental invariant,

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}},$$

which is the familiar curvature from two dimensional Euclidean geometry.

This process can now be repeated, prolonging to ever higher degree and calculating the new invariant at every step.

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Frénet-Serret in \mathbb{R}^2

Let us see how this modern formulation provides us with the classical moving frame in \mathbb{R}^2 .

Calculating the differential of the $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ map $x \rightarrow x \cdot g$ gives

$$\cos(\theta)dx - \sin(\theta)du$$

$$\sin(\theta)dx + \cos(\theta)du.$$

Plugging the moving frame formula

$\theta = -\tan^{-1}(u_x)$, $a = x$, $b = u$ into these equations gives the following coframe on \mathbb{R}^2

$$\frac{1}{\sqrt{1+u_x^2}}(dx + u_x du)$$

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The dual vectors to this coframe form the classical moving frame

$$\frac{1}{\sqrt{1+u_x^2}}\left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u}\right),$$
$$\frac{1}{\sqrt{1+u_x^2}}\left(-u_x \frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right).$$

Equi-affine geometry

As another example of a moving frame, consider the action of the special affine group $SA(2) = SL(2) \ltimes \mathbb{R}^2$ on \mathbb{R}^2 via

$$(x, u) \rightarrow (\delta(x-a) - \beta(u-b), -\gamma(x-a) + \alpha(u-b)), \quad \alpha\delta - \beta\gamma = 1.$$

The first few prolongations of this action are

$$u_x \rightarrow -\frac{\gamma - \alpha u_x}{\delta - \beta u_x}, \quad u_{xx} \rightarrow -\frac{u_{xx}}{(\delta - \beta u_x)^3},$$

$$u_{xxx} \rightarrow -\frac{(\delta - \beta u_x)u_{xxx} + 3\beta u_{xx}^2}{(\delta - \beta u_x)^5},$$

$$u_{xxxx} \rightarrow -\frac{(\delta - \beta u_x)^2 u_{xxxx} + 10\beta(\delta - \beta u_x)u_{xx}u_{xxx} + 15\beta^2 u_{xx}^3}{(\alpha + \beta u_x)^7}.$$

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Normalizing the parameters

This action is free on $J^{(3)} = \mathbb{R}^5$. Let

$\mathcal{K} = \{x = u = u_x = 0, u_{xx} = 1, u_{xxx} = 0\}$ and solve the following equations for the parameters $\alpha, \beta, \gamma, \delta, a, b$,

$$0 = \delta(x - a) - \beta(u - b), \quad 0 = -\gamma(x - a) + \alpha(u - b)$$

$$0 = -\frac{\gamma - \alpha u_x}{\delta - \beta u_x}, \quad 1 = -\frac{u_{xx}}{(\delta - \beta u_x)^3},$$

$$0 = -\frac{(\delta - \beta u_x)u_{xxx} + 3\beta u_{xx}^2}{(\delta - \beta u_x)^5}.$$

The solution to the above system is

$$\alpha = \sqrt[3]{u_{xx}}, \quad \beta = -\frac{1}{3}u_{xx}^{-5/3}u_{xxx},$$

$$\gamma = u_x \sqrt[3]{u_{xx}}, \quad \delta = u_{xx}^{-1/3} - \frac{1}{3}u_{xx}^{-5/3}u_{xxx},$$

$$a = x, \quad b = u.$$

Equi-affine curvature

Invariantizing the fourth jet coordinate

$$u_{xxxx} \rightarrow -\frac{(\delta - \beta u_x)^2 u_{xxxx} + 10\beta(\delta - \beta u_x) u_{xx} u_{xxx} + 15\beta^2 u_{xx}^3}{(\alpha + \beta u_x)^7},$$

by plugging into its formula the newly found moving frame, gives a the differential invariant

$$\kappa = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{3u_{xxx}^{8/3}}.$$

This is the well known *equi-affine curvature*.

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Invariant operators

Using the remarkable theory of *contact forms*, which is a fundamental ingredient in the geometric theory of PDEs, we can find so-called *invariant differential operators* that generate the full algebra of differential invariants through repeatedly acting on finitely many fundamental invariants.

A differential operator, \mathcal{D} , is called invariant, if $\mathcal{D}(K)$ is invariant for any invariant K .

In practice, such operators are found by invariantizing the *horizontal coframe* on the space of independent variables (the X in $M = X \times U$) and taking its *dual frame*.

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Arc-length

Going back to the action of $SE(2)$ on \mathbb{R}^2 there is only the one horizontal form dx .

Invariantizing it gives a one form $\iota(dx) = ds = \sqrt{1 + u_x^2} dx$. This is the familiar arc-length element in Euclidean geometry.

Differentiating with respect to the dual of the arc-length element corresponds to differentiating with respect to arc length. The differential operator is given by

$$\mathcal{D} = \frac{1}{\sqrt{1 + u_x^2}} D_x.$$

Now, every differential invariant can be obtained by repeated application of the invariant \mathcal{D} to the curvature κ . For example, the third order invariant is

$$\mathcal{D}(\kappa) = \frac{d\kappa}{ds} = \frac{u_{xxx}(1 + u_x^2)^{3/2} - \frac{3}{2}(1 + u_x^2)^{1/2} \cdot 2u_x u_{xx}^2}{(1 + u_x^2)^7}.$$

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The differential invariant algebra

The collection of differential invariants and the invariant differential operators form a so-called D-module, and their algebraic structure is of great interest.

The Fundamental Basis Theorem says that for any free and regular action on M , the entire differential algebra of invariants can be generated by a finite set of fundamental invariants through repeated invariant differentiation. This assumes the role of *The Hilbert Basis Theorem* for polynomial ideals.

An open problem is the determination of a *minimal generating set* of differential invariants. There is some hope that Gröbner bases techniques in conjunction with results on syzygy classification based on the moving frame, can give insight into this problem.

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QUESTIONS?